Mathematical Tools for Asset Management MTH6113

Stochastic Dominance

Measures of Investment Risk

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Spring Term

Plan

- Limitations of Utility Theory
- Stochastic Dominance
  - First Order Stochastic Dominance
  - Second Order Stochastic Dominance
- Introduction to Risk and Return

Limitations of Utility Theory

1. It may not be possible to model investor's behaviour over all possible levels of wealth with a single utility function

- quadratic utility function only satisfies non-satiation for a limited wealth range
  - if there is discontinuous change in investor's preferences at certain levels of wealth
    - use **state-dependent utility functions:**
    - utility of an umbrella depends on the weather
    - risk appetite of an insurance company may depend on whether it is still solvent/has just become insolvent

Limitations of Utility Theory

2. Typically utility of investor not known
3. Difficult to apply to several choices
4. Difficult to apply to a firm with many sources of risk and many views on that risk
5. Utility theory not an efficient mechanism for modelling interdependencies of different risks

Stochastic Dominance

Question: How can we rank investments/gambles/lotteries if

- **we don't know the exact individual utility function**
- **but we know the distribution of returns of investment**

Answer: Use **Stochastic dominance** or relative outperformance of one investment vs. another one

- reminder: cdf of portfolio i which yields wealth x less than or equal to L

$$F_i\left(L\right) = \int_{-\infty}^{L} f_i\left(x\right) dx = \Pr\left(x \leq L\right) \text{ continuous case}$$

$$F_i\left(L\right) = \sum_{x \leq L} f_i\left(x\right) = \Pr\left(x \leq L\right) \text{ discrete case}$$

First Order Stochastic Dominance

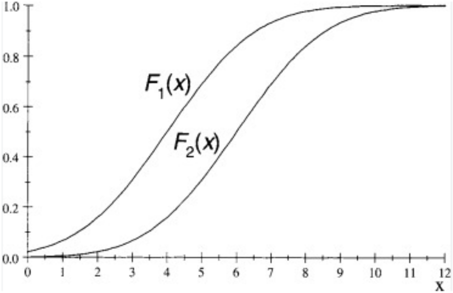
Portfolio A **first order dominates** B (the investor will prefer Portfolio A to Portfolio B) if:

$$\begin{aligned} F_A(x) &\leq F_B(x) \text{ for all } x \text{ and} \\ F_A(x) &< F_B(x) \text{ for some } x \end{aligned}$$

Probability of B producing a return below a certain value is never less than the probability of Portfolio A producing a return below the same value and exceeds it at least once

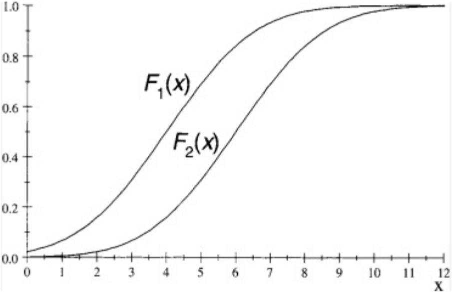
First Order Stochastic Dominance

Asset 1 or Asset 2 first order stochastically dominates the other?



First Order Stochastic Dominance

Asset 1 or Asset 2 first order stochastically dominates the other?



Asset 2 FOSD Asset 1

First Order Stochastic Dominance

Example

Which investment first order dominates the other?

Asset X		Asset Z	
return	probability	return	probability
5%	0.5	6%	0.5
7%	0.5	8%	0.5

First Order Stochastic Dominance

	X	Z	X		Z
Returns	*pdf*	*pdf*	*cdf*		*cdf*
5%	0.5	0	0.5	$>$	0
6%	0	0.5	0.5	$=$	0.5
7%	0.5	0	1	$>$	0.5
8%	0	0.5	1	$=$	1

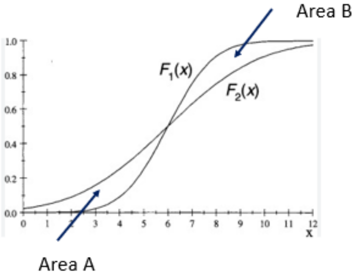
Hence Z FOSD X

Second Order Stochastic Dominance

Portfolio A **second order dominates** B if:

$$\begin{aligned} \int_a^x F_A(y)\,dy &\leq \int_a^x F_B(y)\,dy \text{ for all } x \text{ and} \\ \int_a^x F_A(y)\,dy &< \int_a^x F_B(y)\,dy \text{ for some } x \end{aligned}$$

Second Order Stochastic Dominance



Asset 1 SOSD Asset 2:

- as long as area A is strictly greater than Area B
- *a potential gain of a certain amount is not valued as highly as a loss of the same amount by a risk averse investor*

Example

Which investment second order dominates the other?

	Asset X	Asset Z
return	probability	probability
6%	0.25	0
7%	0.25	0.75
8%	0.25	0
9%	0.25	0.25

Second Order Stochastic Dominance

	X	Z	X		Z	X		Z
Returns	*pdf*	*pdf*	*cdf*		*cdf*	$\sum cdf$		$\sum cdf$
6%	0.25	0	0.25	$>$	0	0.25	$>$	0
7%	0.25	0.75	0.5	$<$	0.75	0.75	$=$	0.75
8%	0.25	0	0.75	$=$	0.75	1.5	$=$	1.5
9%	0.25	0.25	1	$=$	1	2.5	$=$	2.5

Hence Z SOSD X

Measures of Investment Risk

Question: How can we rank investments/gambles/lotteries if

- if **we don't know the whole distribution of returns of investment/asset**

Answer: Use **partial known information on the distribution of returns** (i.e. moments of distribution)

Return on Asset: percentage increase in the market value of an asset over a specified period

- Discrete Random Variable X can take values $x_{1,}..., x_n$ with probabilities $p_1..., p_n$ and $\sum_i p_i = 1$
- Continuous Random Variable X can take values across a range characterised by a p.d.f. $f(x)$

Measures of Investment Risk

- Expected Value/Mean

$$E(X) \equiv \mu = \sum_i p_i x_i \text{ if } X \text{ is discrete}$$

$$E(X) \equiv \mu = \int_{-\infty}^{\infty} x f(x) dx \text{ if } X \text{ is continuous}$$

- Measures of investment risk:
  - variance of return
  - downside semi-variance of return
  - shortfall probabilities
  - value at risk (next week)
  - expected shortfall (next week)

Variance of Return

Most theories of investment risk use variance of return as the measure of risk

$$Var\left(X\right) \equiv \sigma^2 = \sum_i (x_i - \mu)^2 p_i \text{ if } X \text{ is discrete}$$

$$Var\left(X\right) \equiv \sigma^2 = \int_{-\infty}^{\infty} \left(x - \mu\right)^2 f\left(x\right) dx \text{ if } X \text{ is continuous}$$

Example

The investment annual returns X for a particular stock are modelled using a pdf:

$$\begin{aligned} f(x) &= 750\left(0.01 - (x-0.05)^2\right) \\ -0.05 &\le x \le 0.15 \text{ or } -5\% \le x \le 15\% \end{aligned}$$

Verify that $f(x)$ is a proper pdf:

$$\begin{aligned} \int_{-0.05}^{0.15} f(x)\,dx &= \int_{-0.05}^{0.15} 750\left(0.01 - (x-0.05)^2\right) dx \\ &= 750 \int_{-0.05}^{0.15} \left(0.0075 + 0.1x - x^2\right) dx \\ &= 750 \left[0.0075x + \frac{0.1x^2}{2} - \frac{x^3}{3}\right]_{-0.05}^{0.15} \\ &= 1 \end{aligned}$$

Example

The average of the returns:

$$
\begin{aligned}
E\left(X\right) &= 750\int_{-0.05}^{0.15} x\left(0.01-\left(x-0.05\right)^2\right)dx \\
&= 750\int_{-0.05}^{0.15} x\left(0.01-\left(x^2-0.1x+0.0025\right)^2\right)dx \\
& \ 750\int_{-0.05}^{0.15} \left(0.0075x+0.1x^2-x^3\right)dx \\
&= 750\left[\frac{0.0075}{2}x^2+\frac{0.1x^3}{3}-\frac{x^4}{4}\right]_{-0.05}^{0.15} = 0.05
\end{aligned}
$$

Example

The variance of the returns for the same stock:

$$\begin{aligned} Var\left(X\right) &= \int_{-0.05}^{0.15} 750\left(0.01-\left(x-0.05\right)^2\right)\left(x-0.05\right)^2 dx \\ &= 750\int_{-0.05}^{0.15}\left(0.01\left(x-0.05\right)^2-\left(x-0.05\right)^4\right) dx \\ &= 750\left[\frac{0.01}{3}\left(x-0.05\right)^3-\frac{1}{5}\left(x-0.05\right)^5\right]_{-0.05}^{0.15} = 0.002 \end{aligned}$$

Semi-Variance of Return

The *(downside) semi-variance* of return (SV) is defined as:-

$$SV(X) = \sum_{x_i \le \mu} (x_i - \mu)^2 p_i \text{ if } X \text{ is discrete}$$

$$SV(X) = \int_{-\infty}^{\mu} (x - \mu)^2 f(x) dx \text{ if } X \text{ is continuous}$$

- It doesn't take into account the variability above the mean ('upside risk')
- It is not so easy to handle mathematically
- How does this relate to variance?

Example

Continuing the example:

$$
\begin{aligned}
SV(X) &= \int_{-0.05}^{0.05} 750\left(0.01-(x-0.05)^2\right)(x-0.05)^2\,dx \\
&= 750\int_{-0.05}^{0.05}\left(0.01\,(x-0.05)^2-(x-0.05)^4\right)dx \\
&= 750\left[\frac{0.01}{3}(x-0.05)^3-\frac{1}{5}(x-0.05)^5\right]_{-0.05}^{0.5} = 0.001
\end{aligned}
$$

Shortfall Probabilities

A shortfall probability (SP) measures the probability of returns falling below a certain level - the risk of ruin:

$$\sum_{x<L} p_i \text{ if } X \text{ is discrete}$$

$$\int_{-\infty}^{L} f(x)dx \text{ if } X \text{ is continuous}$$

- L: the chosen benchmark level
  - an absolute level required to meet a payment
  - return on a benchmark fund

Example

Continuing the example, find the shortfall probability for the stock given that the benchmark return is 0.

$$
\begin{aligned}
SP(X) &= \Pr(X < 0) = \\
&= \int_{-0.05}^{0} 750\left(0.01 - (x - 0.05)^2\right) dx \\
&= 750\left[0.01x - \frac{(x - 0.05)^3}{3}\right]_{-0.05}^{0} = 0.15625
\end{aligned}
$$