MTP24, Lecture 4: Martingales (continued), Convergence and the Brownian Motion

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Stopping times

Definition *Stopping time* adapted to filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ is a r.v. τ with values in $\{0, 1, \cdots, \infty\}$, s.t.

$$\{\tau = n\} \in \mathcal{F}_n, \quad n = 0, 1, \ldots$$

If $\tau < \infty$ a.s. the stopping time is called *finite*.

For $\xi_0, \xi_1, \ldots$ with natural filtration, examples of stopping times are $\tau = \min\{n : \xi_n > c\}, \tau = \min\{n > 0 : \xi_n > \xi_0\}$ etc.

The stopped variable is defined as

$$\xi_\tau = \sum_{n=0}^{\infty} \xi_n \mathbf{1}_{\{\tau = n\}}$$

and the stopped process as

$$\xi_{\tau \wedge n}, \ n \geq 0.$$

Proposition If (X_n) is a martingale (sub-, super-) then $(X_{\tau \wedge n}, \ n \geq 0)$ is a martingale (sub-, super-) too.

Doob's optional sampling

Theorem Let (X_n) be supermartingale, τ stopping time. Then

$$\mathbb{E}[X_\tau] \leq \mathbb{E}[X_0]$$

if at least one of the following holds:

(i) $\mathbb{P}[\tau < K] = 1$ for some $K > 0$,

(ii) $\sup_n |X_n| < K$ a.s.

(iii) $\mathbb{E}[\tau] < \infty$ and $\sup_n \mathbb{E}\left[|X_{n+1} - X_n| \,|\mathcal{F}_n\right] < K$,

(iv) (X_n) is uniformly integrable.

First passage time for a RW

For (S_n) symmetric ± 1-random walk with $S_0 = 0$, consider $\tau = \min\{n : S_n = 1\}$. Since $\mathbb{E}[\exp(zS_n) = (\cosh z)^n$

$$M_n = \frac{\exp(zS_n)}{(\cosh z)^n}$$

is a martingale. By the Optional Sampling

$$\mathbb{E}\left[\frac{\exp(zS_\tau)}{(\cosh z)^\tau}\right] = 1,$$

where $S_\tau = 1$. Changing variable to $x = (\cosh z)^{-1}$ gives

$$\mathbb{E}[x^\tau] = \frac{1 - \sqrt{1 - x^2}}{x},$$

whence the distribution of τ is

$$\mathbb{P}[\tau = 2m - 1] = (-1)^{m+1}\binom{1/2}{m}.$$

Wald's identities

If $\xi_1, \xi_2, \ldots$ i.i.d. with finite $\mu = \mathbb{E}[\xi_1], \sigma^2 = \mathrm{Var}[\xi_1]$, then for $S_n = \sum_{i=1}^n \xi_n$ it holds that

$$\begin{aligned} \mathbb{E}[S_\tau] &= \mu\,\mathbb{E}[\tau], \\ \mathbb{E}[S_\tau - \tau\,\mu]^2 &= \sigma^2\,\mathbb{E}[\tau]. \end{aligned}$$

Martingale convergence

Theorem Let $(X_n, n \geq 0)$ be a submartingale with $\sup_n \mathbb{E}|X_n| < \infty$. Then the exists a r.v. X_∞ with $\mathbb{E}|X_n| < \infty$ such that

$$X_n \to X_\infty \text{ as } n \to \infty \text{ a.s.}$$

If the uniform integrability condition holds

$$\lim_{c \to \infty} \sup_n \mathbb{E}\left[|X_n| \mathbf{1}(|X_n| > c)\right] = 0,$$

then also $\mathbb{E}|X_n - X_\infty| \to 0$.

Example (Doob martingale) If $\mathbb{E}|\xi| < \infty$ then for $\mathcal{F}_\infty := \sigma\left(\cup_n \mathcal{F}_n\right)$

$$\mathbb{E}[\xi|\mathcal{F}_n] \to \mathbb{E}[\xi|\mathcal{F}_\infty] \quad \text{a.s.}$$

Application: supercritical branching process

ξ_{nj} i.i.d. $\mathbb{Z}_+$-valued r.v.'s with $\mu := \mathbb{E}[\xi_{ni}] > 1$. The Galton-Watson branching process has $Z_0 = 1$ and

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{ni}.$$

By Wald's identity

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mu Z_n, \quad \mathcal{F}_n = \sigma(Z_0, \ldots, Z_n),$$

hence we have a martingale

$$\frac{Z_n}{\mu^n}, \quad n \geq 0,$$

which by the Martingale Convergence has a limit

$$\frac{Z_n}{\mu^n} \to \quad W \quad \text{a.s.}$$

with $\mathbb{P}[W = 0] = \mathbb{P}[\cup_n \{Z_n = 0\}]$ being the probability of extinction.

A random process $(C_n, n \geq 0)$ is *predictable* if C_n is $\mathcal{F}_{n-1}$-measurable ($\mathcal{F}_{-1} = \mathcal{F}_0$

Definition Let $(X_n, n \geq 0)$ be martingale, $(C_n, n \geq 1)$ predictable. The *martingale transform* is

$$Y_n = C_0 X_0 + \sum_{k=1}^{n} C_k (X_k - X_{k-1}), \quad n = 1, 2, \ldots$$

The martingale transform satisfies

$$Y_n = \mathbb{E}[Y_{n+1} | \mathcal{F}_n],$$

but in general $\mathbb{E}|Y_n| < \infty$ may fail (*generalised* martingale).

Doob-Meyer decomposition of submartingale

For submartingale (X_n) there exists a unique representation

$$X_n = M_n + C_n,$$

where (M_n) a martingale and (C_n) a predictable process. Explicitly,

$$M_n = X_0 + \sum_{k=0}^{n-1} \left(X_{k+1} - \mathbb{E}[X_{k+1}|\mathcal{F}_k] \right),$$

and

$$C_n = \sum_{k=1}^{n-1} \left(\mathbb{E}[X_{k+1}|\mathcal{F}_k] - X_k \right).$$

Example Biased RW (S_n) with $p > 1$: $M_n = S_n - (2p - 1)n$.

Quadratic characteristic

Let (X_n) be a martingale with $\mathrm{Var}[X_n] < \infty$. The submartingale $X_n^2, n \geq 0$ decomposes as

$$X_n^2 = M_n + \langle X \rangle_n,$$

where

$$\langle X \rangle_n := \sum_{k=1}^{n} \mathbb{E}\left[(X_k - X_{k-1})^2 | \mathcal{F}_k\right]$$

is the *quadratic characteristic* of (X_n), which satisfies

$$\mathbb{E}\left[(X_n - X_m)^2 | \mathcal{F}_m\right] = \mathbb{E}[\langle X \rangle_n - \langle X \rangle_m | \mathcal{F}_m].$$

Example Let ξ_n be independent, $\mathbb{E}[\xi_n] = 0, \mathrm{Var}[\xi_n] = \sigma_n^2 < \infty$, then $S_n = \xi_1 + \ldots + \xi_n, n \geq 0$, is a martingale with the quadratic characteristic

$$\langle S \rangle_n = \mathrm{Var}[S_n] = \sigma_1^2 + \cdots + \sigma_n^2.$$

Maximal inequalities

For $(X_n, \mathcal{F}_n)$ submartingale, $c > 0$,

$$\mathbb{P}[\max_{k \leq n} X_k \geq c] \leq \frac{\mathbb{E}[\, X_n^+ \,]}{c},$$

and for martingale

$$\mathbb{P}[\max_{k \leq n} |X_k| \geq c] \leq \frac{\mathbb{E}[\, |X_n|^2 \,]}{c^2}.$$

Convergence modes

$(X_n,\ n \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ converges to X

- *almost surely* if $\mathbb{P}[X_n \to X] = 1$,
- $X_n \stackrel{\mathbb{P}}{\to} X$ (*in probability*) if $\lim_{n\to\infty} \mathbb{P}[|X_n - X| > \epsilon) = 0 \ \ \forall \epsilon > 0$,
- $X_n \stackrel{L^p}{\to} X$ (*in pth mean*, $p > 0$) if $\lim_{n\to\infty} \mathbb{E}[|X_n - X|^p|] = 0$.
- $X_n \stackrel{d}{\to} X$ *in distribution* if $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ for all bounded, continuous $f : \mathbb{R} \to \mathbb{R}$.

Connection:

$$\stackrel{a.s.}{\to} \ \Rightarrow \ \stackrel{\mathbb{P}}{\to},$$
$$\stackrel{L^p}{\to} \ \Rightarrow \ \stackrel{\mathbb{P}}{\to},$$
$$\stackrel{\mathbb{P}}{\to} \ \Rightarrow \ \stackrel{d}{\to}.$$

But

$$\stackrel{\mathbb{P}}{\to} \ \Rightarrow \ \stackrel{a.s.}{\to}$$

along a subsequence. If $X_n \stackrel{d}{\to} X$ then $X'_n \stackrel{\text{a.s.}}{\to} X'$ for some distributional copies $X'_n \stackrel{d}{=} X_n, X' \stackrel{d}{=} X$.

Weak convergence

Let P and $P_n, n \in \mathbb{N}$, be probability measures on a metric space E (with Borel σ-algebra).

Definition $P_n \xrightarrow{w} P$, that is P_n *converge weakly* to P, if

$$\int_E f(x) P_n(\mathrm{d}x) \to \int_E f(x) P(\mathrm{d}x)$$

for all bounded, continuous functions $f : E \to \mathbb{R}$.

Equivalently, $P_n \xrightarrow{w} P$ if any of the following holds true:

(i) $\limsup P_n(A) \leq P(A)$ for closed A,

(ii) $\liminf P_n(A) \geq P(A)$ for open A,

(iii) $P_n(A) \to P(A)$ if $P(\partial A) = 0$, where $\partial A = \mathrm{cl} A \cap \mathrm{cl} A^c$.

The Brownian motion

The BM $(B(t),\ t \geq 0)$ is a continuous-time stochastic process satisfying

(i) $B(0) = 0$ a.s.,

(ii) the path $t \mapsto B(t)$ is continuous a.s.

(iii) the increments $B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$ are independent for any choice $0 \leq t_0 < t_1 < \ldots < t_n$.

(iv) $B(t) - B(s) \stackrel{d}{=} \mathcal{N}(0, t-s),\ 0 \leq s < t$.

Conditions (iii), (iv) can be equivalently replaced by

- $(B(t),\ t \geq 0)$ is Gaussian with $\mathbb{E}[B(t)] = 0$, and

$$\mathrm{cov}(B(s), B(t)) = s \wedge t.$$

Existence of the BM

By Kolmogorov's extension there exists Gaussian $(B(t), t \in \mathbb{Q}_1)$ (where $\mathbb{Q}_1 = \mathbb{Q} \cap [0,1]$) with mean 0 and covariance $s \wedge t$. We need a *uniformly* continuous version on $\mathbb{Q}_1$. Consider the 'modulus of continuity'

$$\Delta_n := \sup_{s,t \in \mathbb{Q}_1 : |s-t| < 1/n} |B(t) - B(s)|,$$

we want $\Delta_n \to 0$ a.s. To estimate this introduce simpler variables

$$Y_{k,n} := \sup_{s,t \in [\frac{k-1}{n}, \frac{k}{n}] \cap \mathbb{Q}_1} |B(t) - B(s)|,$$

then (apply the triangle inequality for the B-values at times $(k-1)/n < s < k/n < t$) we obtain

$$\Delta_n \leq 3 \max_{1 \leq k \leq n} Y_{k,n}.$$

By the stationarity of increments

$$\mathbb{P}[\max_{1 \leq k \leq n} Y_{k,n} \geq \epsilon] \leq \sum_{k=1}^{n} \mathbb{P}[Y_{k,n} \geq \epsilon] = n\,\mathbb{P}[Y_{1,n} \geq \epsilon].$$

$(B(t), t \in \mathbb{Q})$ martingale $\Rightarrow$ $(B^4(t), t \in \mathbb{Q})$ submartingale, and the maximal inequality gives

$$n\,\mathbb{P}[Y_{1n} \geq \epsilon] = n\,\mathbb{P}\left[\max_{t \in \mathbb{Q} \cap [0,1/n]} |B(t)| \geq \epsilon\right] \leq$$
$$\frac{n}{\epsilon^4}\,\mathbb{E}\left[B^4(1/n)\right] = \frac{3n}{n^2\epsilon^4} \to 0 \qquad \text{as } n \to \infty,$$

which gives $\Delta_n \stackrel{\mathbb{P}}{\to} 0$ but since $\Delta_1 \geq \Delta_2 \geq \cdots$ a.s. also $\Delta_n \stackrel{\text{a.s.}}{\to} 0$.

$\Rightarrow$ the BM extends from $t \in \mathbb{Q}_1$ to $[0, 1]$ by continuity.

Further properties of the BM

• BM is nowhere differentiable,

• the variation is infinite on any interval (same holds for the length of Brownian path),

• the quadratic variation is $\langle B \rangle(t) = t,\ t \geq 0$,

Hölder continuity with exponent $0 < \alpha < 1/2$

$$\sup_{s,t \in [0,1]} |B(t) - B(s)| < C\, |t - s|^{\alpha} \quad \text{a.s.}$$

• the set of zeroes $\{t : B(t) = 0\}$ is a.s. closed without isolated points.