MTH5131 Actuarial Statistics

Coursework 2

This coursework is not to be turned in. You may ask questions about the coursework in tutorial or by email.

Exercise 1. Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent $\mathrm{Bin}(m, \pi)$ random variables, where m is known.

1. Write down the likelihood for the data $y_1, y_2, \ldots, y_n$.

2. Show that $\overline{Y}/m$ is an unbiased estimator of π and find its variance. What is the value for c such that $c\overline{Y}(m - \overline{Y})/m^2$ is an unbiased estimator of this variance?

Exercise 2. Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent $\mathrm{N}(\mu, \sigma^2)$ random variables.

1. Show that $\overline{Y}$ is an unbiased estimator of μ.

2. Is $\overline{Y}$ consistent for μ?

3. Show that $\sum_{i=1}^{n}(Y_i - \overline{Y})^2/n$ is a biased estimator of σ^2.

Exercise 3. Let $X \sim \mathrm{Binomial}(n, \theta)$, where n is known, and we want to estimate θ. Consider the estimator of θ given by $T_n = \frac{X+1}{n+2}$.

1. Find $\mathrm{bias}(T_n)$ and show that each T_n is an biased estimator of θ.

2. Find the MSE of T_n for θ.

3. Show that T_n is a consistent sequence of estimators for θ.

Exercise 4. Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent random variables from the beta distribution with probability density function

$$f_Y(y) = \frac{1}{\theta} y^{\frac{1}{\theta}-1}, \quad 0 < y < 1,$$

where $\theta > 0$.

1. Write down the likelihood for the data $y_1, y_2, \ldots, y_n$.
2. Show that $E(\log Y) = -\theta$. Hence, show that $-\sum_{i=1}^{n} \log(Y_i)/n$ is an unbiased estimator of θ.
3. Find the Cramér-Rao lower bound for unbiased estimators of θ.
4. Given that $\text{var}(\log Y) = \theta^2$, is $-\sum_{i=1}^{n} \log(Y_i)/n$ a minimum variance unbiased estimator of θ?

Exercise 5. Let $Y_1, Y_2, \ldots, Y_n$ be independent lognormal random variables with probability density function

$$f_Y(y) = \frac{1}{\sqrt{2\pi} y} \exp\left\{-\frac{1}{2}(\log y - \theta)^2\right\}, \quad y > 0,$$

where $-\infty < \theta < \infty$. Given that $E(Y) = \exp(\theta + 1/2)$, obtain the method of moments estimator of θ.

Exercise 6. Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent lognormal random variables with probability density function

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2} y} \exp\left\{-\frac{1}{2\sigma^2}(\log y - \mu)^2\right\}, \quad y > 0,$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$ are parameters. Given that

$$E(Y) = e^{\mu + \frac{\sigma^2}{2}} \quad \text{and} \quad E(Y^2) = e^{2\mu + 2\sigma^2},$$

find the method of moments estimators for μ and σ^2.

Exercise 7. Let $Y_1, Y_2, \ldots, Y_n$ be independent $\text{Bin}(m, \pi)$ random variables, where m is known.

1. Show that the method of moments estimator of π is $\overline{Y}/m$.
2. Find the maximum likelihood estimator of π.
3. The Fisher information of π for $Y_1, Y_2, \ldots, Y_n$ is given by[1]

$$I(\pi) = \frac{nm}{\pi(1-\pi)}.$$

Write down the asymptotic distribution of the maximum likelihood estimator.

[1] As a further exercise, try calculating this.

Exercise 8. Suppose that $Y_1, Y_2, \dots, Y_n$ are independent random variables with mean $\theta + 1$ and probability density function

$$f_Y(y) = e^{-(y-\theta)}, \quad y \geq \theta,$$

where $-\infty < \theta < \infty$.

1. Explain why the maximum likelihood estimator of θ is $\hat{\theta} = \min_i Y_i$.

2. Find the method of moments estimator of θ.

Exercise 9.

A random variable X with Type 1 negative binomial distribution having parameters p and k counts the number of the trial on which the kth success occurs, where k is a positive integer It has p.d.f.

$$P(X = x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, \quad x = k, k+1, \dots$$

Another formulation of the negative binomial distribution is sometimes used, which is called the Type 2 negative binomial distribution . Let Y be the number of failures before the kth success.o Then,

$$P(Y = y) = \binom{k+y-1}{y} p^k (1-p)^y, \quad y = 0, 1, \dots$$

with expectation $E(Y) = \frac{k(1-p)}{p}$ and variance var$(X) = \frac{k(1-p)}{p^2}$. Note that $Y = X - k$, where X is defined as above.

A random sample of eight observations from a distribution are given below:

$$4.8 \quad 7.6 \quad 1.2 \quad 3.5 \quad 2.9 \quad 0.8 \quad 0.5 \quad 2.3$$

Derive the method of moments estimates for:

1. λ from an Exponential(λ) distribution

2. ν from a χ^2_ν distribution.

3. k and p from a Type 2 negative binomial distribution. What do you notice about this estimate?

Exercise 10. A discrete random variable has probability density function given by

$$P(X = 2) = \frac{1}{8} + 2\alpha, \quad P(X = 4) = \frac{1}{2} - 3\alpha, \quad P(X = 5) = \frac{3}{8} + \alpha.$$

1. Give the range of possible values for the unknown parameter α.

2. A random sample of 30 observations gave respective frequencies of 7, 6 and 17.

   (a) Calculate the method of moments estimate of α.

   (b) Write down an expression for the likelihood of these data and hence show that the maximum likelihood estimate $\hat{\alpha}$ satisfies the quadratic equation:

   $$180\hat{\alpha}^2 + \frac{111}{8}\hat{\alpha} - \frac{91}{32} = 0.$$

   (c) Hence determine the maximum likelihood estimate and explain why one root is rejected as a possible estimate of α

Exercise 11. Waiting times in a post office queue have an Exponential(λ) distribution. Ten people had waiting times (in minutes) of:

$$1.6 \quad 0.9 \quad 1.1 \quad 2.1 \quad 0.7 \quad 1.5 \quad 2.3 \quad 1.7 \quad 3.0 \quad 3.4$$

A further six people had waiting times of more than 4 minutes. Calculate the maximum likelihood estimate of λ based on these data.