

# MTH5113 (2023/24): Problem Sheet 1

## Solutions

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(1) (*Warm-up*) Compute each of the following. *Make sure each of your answers is the right type (scalar, vector, tangent vector) of object!*

(a)  $-2 \cdot (-1, 1, 7)_{(2,0,3)} + 3 \cdot (3, -2, 4)_{(2,0,3)}$ .

(b)  $(17, -\frac{11}{3})_{(\pi, e)} \cdot (-\frac{2}{5}, \frac{3}{4})_{(\pi, e)}$ .

(c)  $(1, 1, 1)_{(-1,0,-1)} \times (-1, -1, 1)_{(-1,0,-1)}$ .

(a) First, we compute the scalar products according to definition:

$$\begin{aligned} -2 \cdot (-1, 1, 7)_{(2,0,3)} &= [-2 \cdot (-1, 1, 7)]_{(2,0,3)} = (2, -2, -14)_{(2,0,3)}, \\ 3 \cdot (3, -2, 4)_{(2,0,3)} &= [3 \cdot (3, -2, 4)]_{(2,0,3)} = (9, -6, 12)_{(2,0,3)}. \end{aligned}$$

We then add the resulting tangent vectors:

$$\begin{aligned} -2 \cdot (-1, 1, 7)_{(2,0,3)} + 3 \cdot (3, -2, 4)_{(2,0,3)} &= (2, -2, -14)_{(2,0,3)} + (9, -6, 12)_{(2,0,3)} \\ &= [(2, -2, -14) + (9, -6, 12)]_{(2,0,3)} \\ &= (11, -8, -2)_{(2,0,3)}. \end{aligned}$$

(The answer is a tangent vector, based at  $(2, 0, 3)$ .)

(b) By the definition of the tangent vector dot product,

$$\left(17, -\frac{11}{3}\right)_{(\pi, e)} \cdot \left(-\frac{2}{5}, \frac{3}{4}\right)_{(\pi, e)} = \left(17, -\frac{11}{3}\right) \cdot \left(-\frac{2}{5}, \frac{3}{4}\right).$$

We then compute the right-hand side in the usual way:

$$\left(17, -\frac{11}{3}\right) \cdot \left(-\frac{2}{5}, \frac{3}{4}\right) = -17 \cdot \frac{2}{5} - \frac{11}{3} \cdot \frac{3}{4} = -\frac{191}{20}.$$

(The answer is a scalar, i.e. a real number.)

(c) We first compute the cross product of the directional parts:

$$\begin{aligned}(1, 1, 1) \times (-1, -1, 1) &= (1 \cdot 1 - 1 \cdot (-1), 1 \cdot (-1) - 1 \cdot 1, 1 \cdot (-1) - 1 \cdot (-1)) \\ &= (2, -2, 0).\end{aligned}$$

Thus, by the definition of the tangent vector cross product,

$$\begin{aligned}(1, 1, 1)_{(-1,0,-1)} \times (-1, -1, 1)_{(-1,0,-1)} &= [(1, 1, 1) \times (-1, -1, 1)]_{(-1,0,-1)} \\ &= (2, -2, 0)_{(-1,0,-1)}.\end{aligned}$$

(The answer is a tangent vector, based at  $(-1, 0, -1)$ .)

(2) (*Warm-up*) Simplify (i.e. compute) each of the following expressions, and then plot it on either a Cartesian plane or in 3-dimensional space:

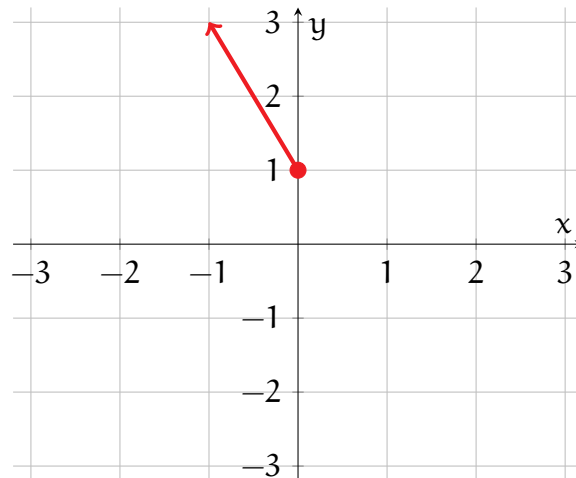
(a)  $3 \cdot (-1, 4)_{(0,1)} - (-2, 10)_{(0,1)}$ .

(b)  $(1, 0, 1)_{(0,1,0)} \times (-1, 1, -1)_{(0,1,0)}$ .

(a) The first part is a direct computation:

$$3 \cdot (-1, 4)_{(0,1)} - (-2, 10)_{(0,1)} = (-3, 12)_{(0,1)} + (2, -10)_{(0,1)} = (-1, 2)_{(0,1)}.$$

This tangent vector is drawn below:

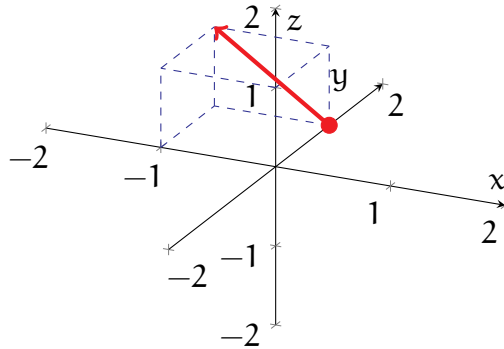


(b) The first part is again a direct computation using the definition of cross product:

$$(1, 0, 1)_{(0,1,0)} \times (-1, 1, -1)_{(0,1,0)} = (0 \cdot (-1) - 1 \cdot 1, 1 \cdot (-1) - 1 \cdot (-1), 1 \cdot 1 - 0 \cdot (-1))_{(0,1,0)}$$

$$= (-1, 0, 1)_{(0,1,0)}.$$

This tangent vector is drawn below (the blue segments are added for perspective):



(3) [Tutorial] The *lemniscate of Geronno* can be represented by the vector-valued function

$$\lambda: \mathbb{R} \rightarrow \mathbb{R}^2, \quad \lambda(t) = (\cos t, \sin t \cos t).$$

(a) Compute  $\lambda(t)$ , for

$$t = 0, \quad t = \frac{\pi}{4}, \quad t = \frac{\pi}{2}, \quad t = \frac{3\pi}{4}, \quad \dots, \quad t = \frac{7\pi}{4}, \quad t = 2\pi.$$

(b) Plot, on a single Cartesian plane, the values  $\lambda(t)$  obtained in part (a).

(c) On the same diagram as in part (b), sketch the image of  $\lambda$ .

(d) Find computer software or a website that can plot parametric curves, and use it to plot the image of  $\lambda$ . (See the *Additional Resources* section on the *QMPlus* page.)

(a) First, we recall the relevant values of cosine and sine:

$$\begin{aligned} \cos 0 &= 1, & \sin 0 &= 0, \\ \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}}, & \sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}}, \\ \cos \frac{\pi}{2} &= 0, & \sin \frac{\pi}{2} &= 1, \\ \cos \frac{3\pi}{4} &= -\frac{1}{\sqrt{2}}, & \sin \frac{3\pi}{4} &= \frac{1}{\sqrt{2}}, \\ \cos \pi &= -1, & \sin \pi &= 0, \end{aligned}$$

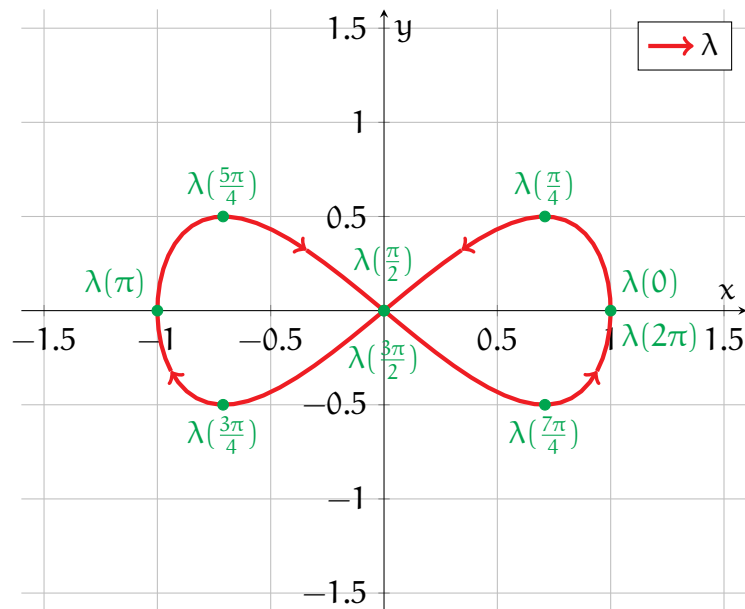
$$\begin{aligned} \cos \frac{5\pi}{4} &= -\frac{1}{\sqrt{2}}, & \sin \frac{5\pi}{4} &= -\frac{1}{\sqrt{2}}, \\ \cos \frac{3\pi}{2} &= 0, & \sin \frac{3\pi}{2} &= -1, \\ \cos \frac{7\pi}{4} &= \frac{1}{\sqrt{2}}, & \sin \frac{7\pi}{4} &= -\frac{1}{\sqrt{2}}, \\ \cos(2\pi) &= 1, & \sin(2\pi) &= 0. \end{aligned}$$

We then use the above to compute the desired values of  $\lambda$ :

$$\begin{aligned} \lambda(0) &= (1, 0), & \lambda\left(\frac{\pi}{4}\right) &= \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), & \lambda\left(\frac{\pi}{2}\right) &= (0, 0), \\ \lambda\left(\frac{3\pi}{4}\right) &= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right), & \lambda(\pi) &= (-1, 0), & \lambda\left(\frac{5\pi}{4}\right) &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \\ \lambda\left(\frac{3\pi}{2}\right) &= (0, 0), & \lambda\left(\frac{7\pi}{4}\right) &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right), & \lambda(2\pi) &= (1, 0). \end{aligned}$$

(b) The values of  $\lambda$  in (a) are drawn as green dots in the figure below.

(c) The image of  $\lambda$  is drawn in red in the figure below.



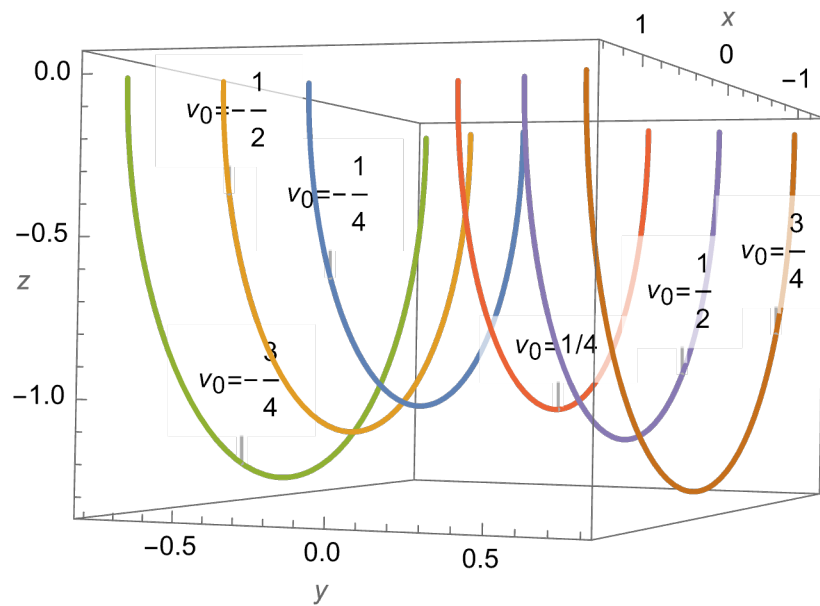
(d) Try it yourself!

(4) [Marked] Consider the following vector-valued function:

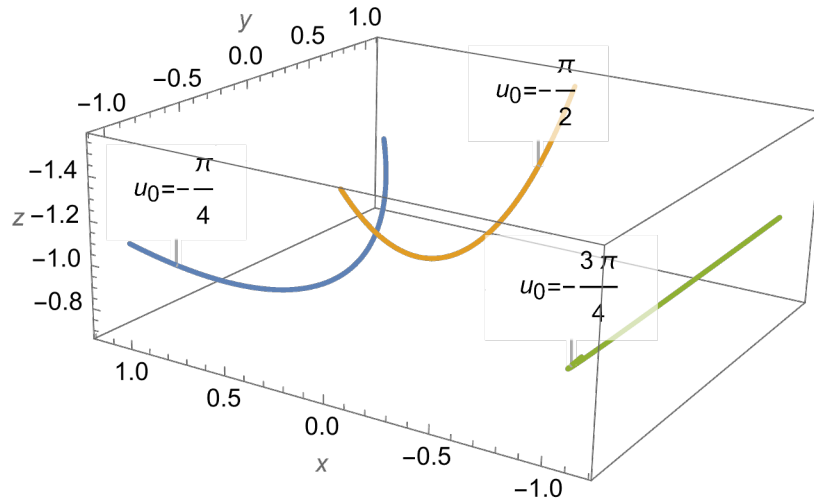
$$\gamma : (-\pi, 0) \times (-1, 1) \rightarrow \mathbb{R}^3, \quad \gamma(\mathbf{u}, \mathbf{v}) = \{\cosh(\mathbf{v}) \cos(\mathbf{u}), \mathbf{v}, \cosh(\mathbf{v}) \sin(\mathbf{u})\}.$$

- (a) Sketch the values of  $\gamma$  obtained by holding  $\mathbf{v} = \mathbf{v}_0$  constant and varying  $\mathbf{u}$ , where (i)  $\mathbf{v}_0 = \pm\frac{1}{4}$ , (ii)  $\mathbf{v}_0 = \pm\frac{1}{2}$ , and (iii)  $\mathbf{v}_0 = \pm\frac{3}{4}$  (you should draw six curves).
- (b) Sketch the values of  $\gamma$  obtained by holding  $\mathbf{u} = \mathbf{u}_0$  constant and varying  $\mathbf{v}$ , where (i)  $\mathbf{u}_0 = -\frac{\pi}{4}$ , (ii)  $\mathbf{u}_0 = -\frac{\pi}{2}$ , and (iii)  $\mathbf{u}_0 = -\frac{3\pi}{4}$ .
- (c) Sketch the full image of  $\gamma$ .

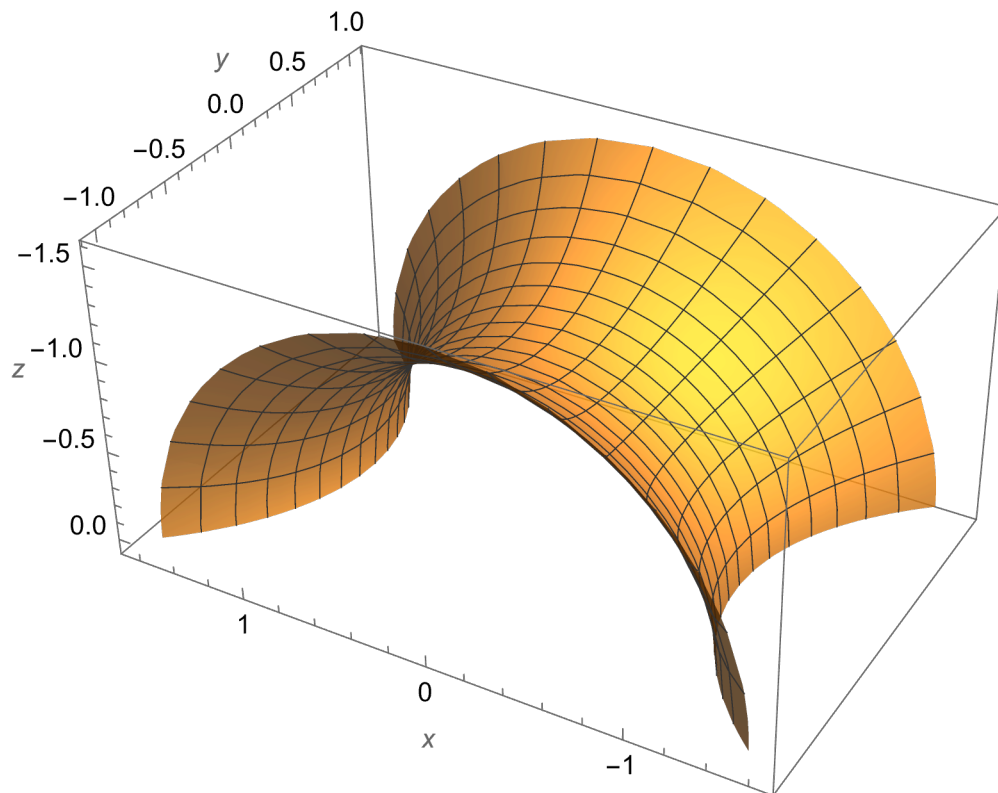
(a) The values obtained by holding  $\mathbf{v}$  constant and varying  $\mathbf{u}$  are in the right figure below: [2 marks for mostly correct drawings (1 mark for partially correct drawings).]



(b) The values obtained by holding  $\mathbf{u}$  constant and varying  $\mathbf{v}$  are in the left figure below: [2 marks for mostly correct drawings (1 mark for partially correct drawings).]



(c) The image of  $\gamma$  is plotted below : [1 mark for mostly correct drawings.]



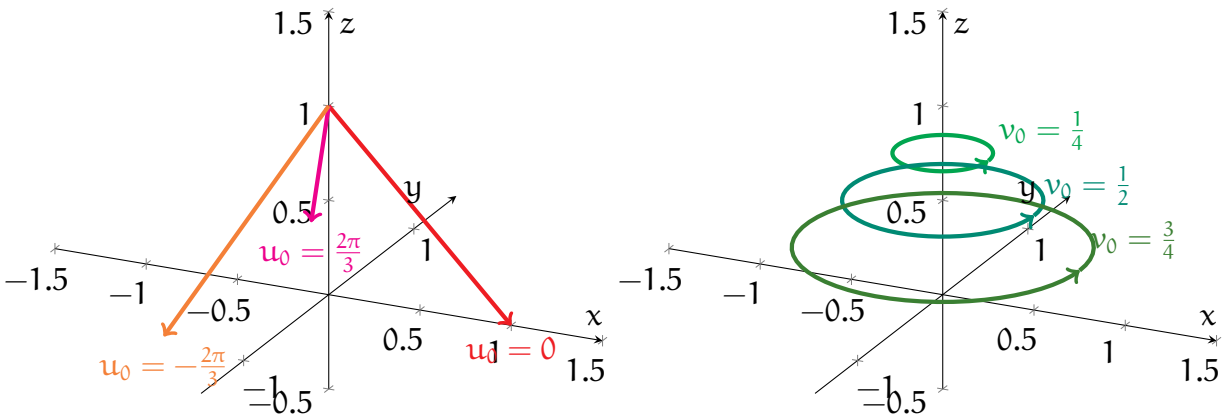
(5) [Tutorial] Consider the following function:

$$\mathbf{f} : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}^3, \quad \mathbf{f}(u, v) = (v \cos u, v \sin u, 1 - v).$$

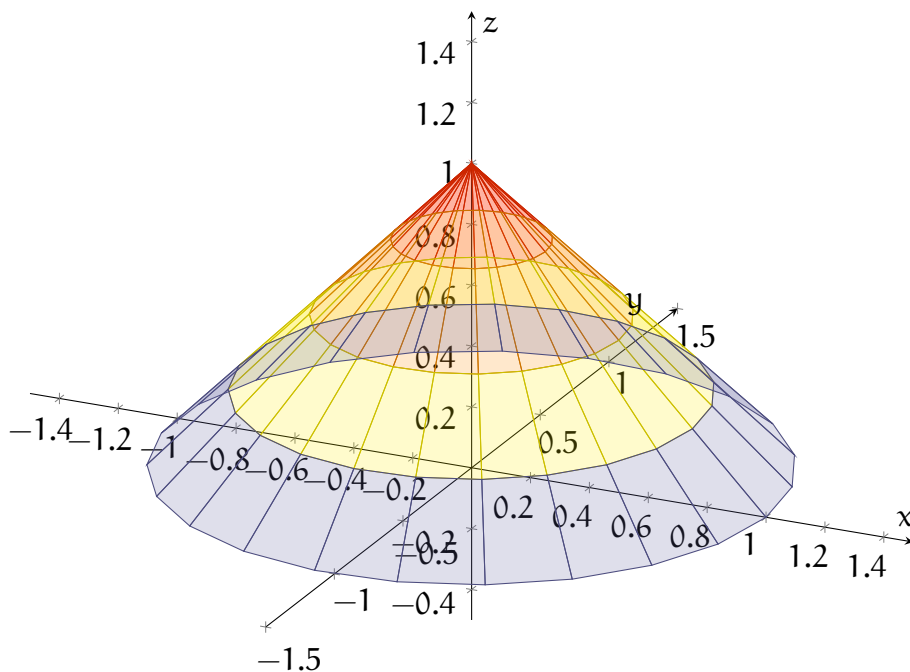
- (a) Sketch the values of  $\mathbf{f}$  obtained by holding  $\mathbf{u} = \mathbf{u}_0$  constant and varying  $\mathbf{v}$ , where (i)  $\mathbf{u}_0 = 0$ , (ii)  $\mathbf{u}_0 = \frac{2\pi}{3}$ , and (iii)  $\mathbf{u}_0 = -\frac{2\pi}{3}$ .
- (b) Sketch the values of  $\mathbf{f}$  obtained by holding  $\mathbf{v} = \mathbf{v}_0$  constant and varying  $\mathbf{u}$ , where (i)  $\mathbf{v}_0 = \frac{1}{4}$ , (ii)  $\mathbf{v}_0 = \frac{1}{2}$ , and (iii)  $\mathbf{v}_0 = \frac{3}{4}$ .
- (c) Sketch the full image of  $\mathbf{f}$ .
- (d) Find computer software or a website that can plot parametric surfaces, and use it to plot the image of  $\mathbf{f}$ . (See the *Additional Resources* section on the *QMPlus* page.)

(a) The values obtained by holding  $\mathbf{u}$  constant and varying  $\mathbf{v}$  are in the left figure below:

(b) The values obtained by holding  $\mathbf{v}$  constant and varying  $\mathbf{u}$  are in the right figure below:



(c) The image of  $\mathbf{f}$  (the side of a cone with circular base) is drawn below:



(d) Try it yourself!

(6) (*You're doing it wrong!*) Mr Error is new to *MTH5113* and has no idea what is going on! Each of the following expressions that Mr Error wrote, while in his confused state, has something very wrong with it—explain the mistake in each expression!

(a)  $-2 \cdot (1, 2)_{(-1, 0)} + 7 \cdot (-2, 4)_{(0, 0)}$ .

(b)  $(2, 7)_{(0, 0)} \times (-1, 6)_{(0, 0)}$ .

(c)  $(2, 4, -1)_{(-1, 2, 3)} + (-3, 1, 1)_{(-1, 2, 3)} \cdot (1, -1, -2)_{(-1, 2, 3)}$ .

(a)  $-2 \cdot (1, 2)_{(-1, 0)}$  and  $7 \cdot (-2, 4)_{(0, 0)}$  denote tangent vectors starting at different points. As a result, it does not make sense to add them—i.e. their sum is not well-defined.

(b) The cross product of 2-dimensional tangent vectors is not defined! (*Don't do this!*)

(c)  $(2, 4, -1)_{(-1, 2, 3)}$  is a tangent vector, while  $(-3, 1, 1)_{(-1, 2, 3)} \cdot (1, -1, -2)_{(-1, 2, 3)}$  is a scalar. Thus, it does not make sense to add these two quantities together.

(7) (*Plotting curves*) For each of the following vector-valued functions  $\gamma$ : (i) plot at least three values of  $\gamma$ , and (ii) sketch the image of  $\gamma$ .



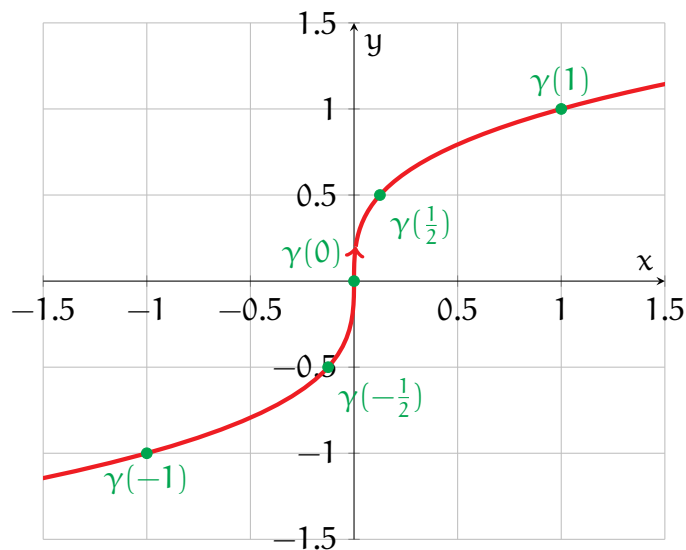
(a)  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ , where  $\gamma(t) = (t^3, t)$ .

(b)  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ , where  $\gamma(t) = (t, \cos(2\pi t), \sin(2\pi t))$ . (It can be a bit tricky to plot a 3-dimensional picture on paper, and you do not need to be too precise here. Do the best you can—the main point is to capture the qualitative behaviour of  $\gamma$ .)

(a) The sketches are given in the figure below.

(i) Several values of  $\gamma$  are drawn in green.

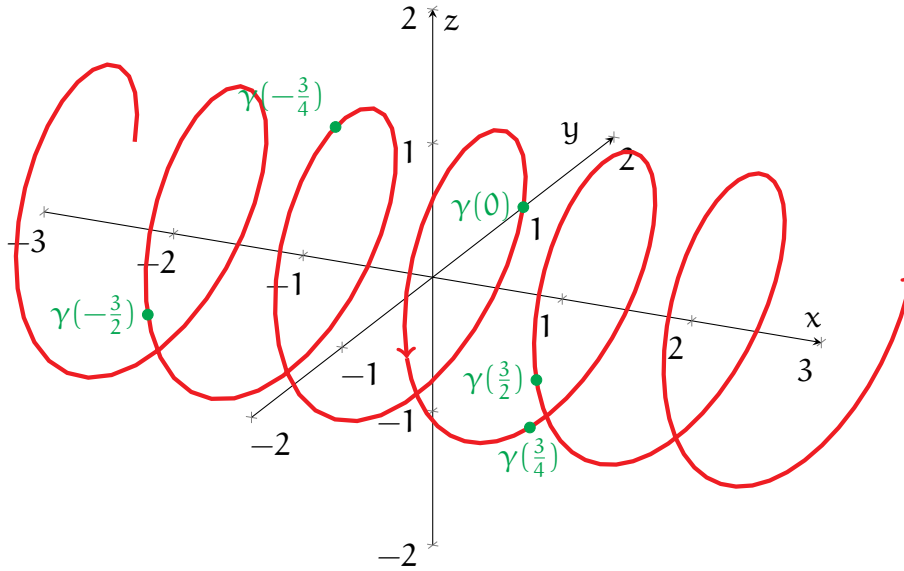
(ii) The image of  $\gamma$  is drawn in red. (It depicts the graph of a cubic function, but with the  $x$  and  $y$ -coordinates interchanged.)



(b) The sketches are given in the figure below.

(i) Several values of  $\gamma$  are drawn in green.

(ii) The image of  $\gamma$  (a *helix*, or coil, wrapping around the  $x$ -axis) is drawn in red.



(8) (*Plotting surfaces*) For each of the following vector-valued functions  $\sigma$ : (i) sketch at least three paths obtained by holding  $\mathbf{u}$  constant and varying  $\mathbf{v}$ , (ii) sketch at least three paths obtained by holding  $\mathbf{v}$  constant and varying  $\mathbf{u}$ , and (iii) sketch the image of  $\sigma$ .

(a)  $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , where  $\sigma(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}, \mathbf{u}^2 + \mathbf{v}^2)$ .

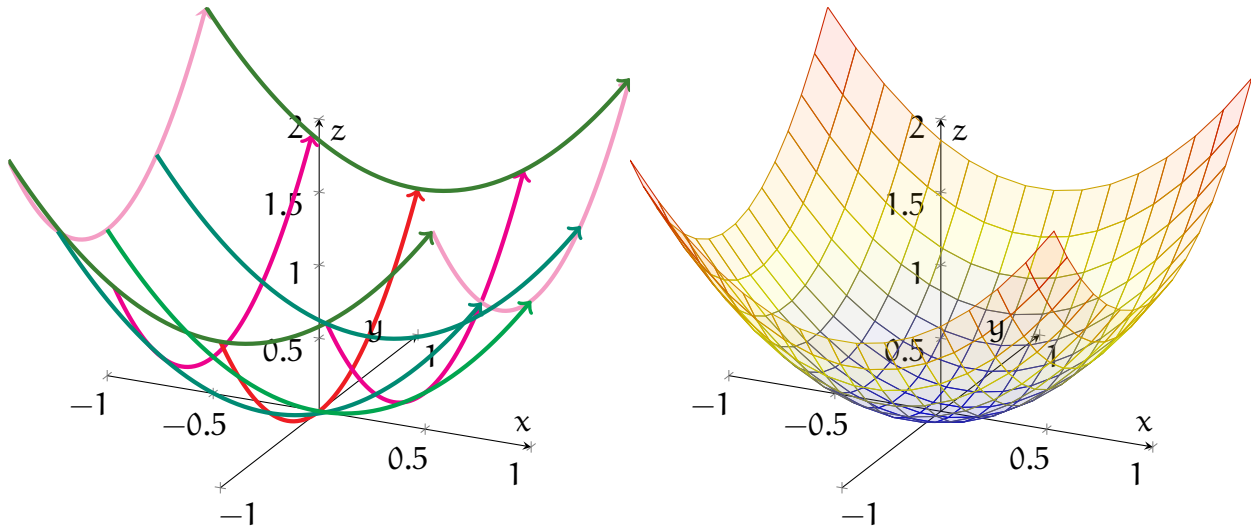
(b)  $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , where  $\sigma(\mathbf{u}, \mathbf{v}) = ((2 + \cos \mathbf{u}) \cos \mathbf{v}, (2 + \cos \mathbf{u}) \sin \mathbf{v}, \sin \mathbf{u})$ .

(a) The sketches are given in the figure below.

(i) The paths of constant  $\mathbf{u}$  and varying  $\mathbf{v}$  are drawn in shades of red in the left plot. Here, we took the constant values  $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$  for  $\mathbf{u}$ .

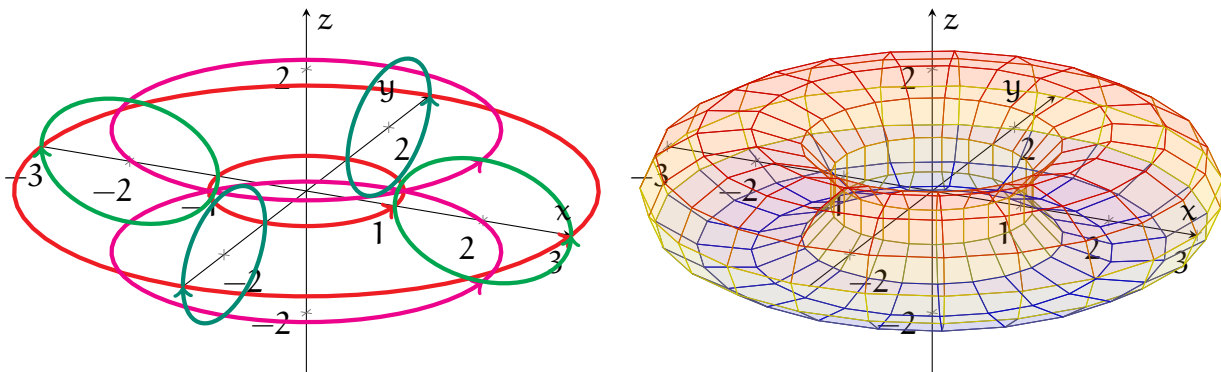
(ii) The paths of constant  $\mathbf{v}$  and varying  $\mathbf{u}$  are drawn in shades of green in the left plot. Here, we took the constant values  $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$  for  $\mathbf{v}$ .

(iii) The image of  $\sigma$  (a *paraboloid*) is drawn in the right plot.



(b) The sketches are in the figure below:

- (i) The paths of constant  $u$  and varying  $v$  are drawn in shades of red in the left plot. Here, we took the constant values  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$  for  $u$ .
- (ii) The paths of constant  $v$  and varying  $u$  are drawn in shades of green in the left plot. Here, we took the constant values  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$  for  $v$ .
- (iii) The image of  $\sigma$  (a *torus*, or “doughnut”) is drawn in the right plot.



(9) (*Fun with Continuity*) Let  $I$  and  $C$  be the open interval and the unit circle

$$I = (0, 1), \quad C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

- (a) Give an informal argument (i.e. a formal proof is not required) as to why there cannot exist an injective and continuous function  $g : C \rightarrow I$ .

(b) Does there exist an injective and continuous function  $\mathbf{h} : I \rightarrow \mathbf{C}$ ?

(a) We can argue by contradiction—suppose  $\mathbf{g} : \mathbf{C} \rightarrow I$  is injective and continuous. Pick any point  $\mathbf{p}_0 \in \mathbf{C}$  (for example,  $\mathbf{p}_0 = (1, 0)$ ), and let  $\mathbf{y}_0 = \mathbf{g}(\mathbf{p}_0) \in (0, 1)$ .

Now, since  $\mathbf{g}$  is injective (so that  $\mathbf{g}$  maps to each value at most once) and continuous (so the values of  $\mathbf{g}$  cannot “jump”), one can see that as one moves anticlockwise from the starting point  $\mathbf{p}_0$ , the corresponding values of  $\mathbf{g}$  must be either strictly increasing or strictly decreasing from the starting value  $\mathbf{y}_0$ . As one travels an entire lap anticlockwise around  $\mathbf{C}$  and returns to the starting point  $(1, 0)$ , the corresponding values of  $\mathbf{g}$  must vary continuously back to  $\mathbf{y}_0 = \mathbf{g}(1, 0)$ . However, this is not possible, since to get back to  $\mathbf{y}_0$ , one must pass again through values of  $\mathbf{g}$  that were already attained earlier.

(b) Yes, this is definitely possible—one simple example is

$$\mathbf{h} : I \rightarrow \mathbf{C}, \quad \mathbf{h}(t) = (\cos(\pi t), \sin(\pi t)),$$

which maps  $I$  (injectively and continuously) onto the upper half of  $\mathbf{C}$ .