MTH5113 (2023/24): Problem Sheet 3

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 1**.

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(1) *(Warm-up)*

(a) Compute the integral

$$\int_0^1 f(x)\, dx,$$

where f is the real-valued function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = 1 + x + x^2 + x^3.$$

(b) Compute the integral

$$\int_{-\pi}^{\pi} g(t)\, dt,$$

where g is the real-valued function

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(t) = \sin t \cos t.$$

(c) Compute the double integral

$$\iint_{\mathcal{R}} h\, dA,$$

where $\mathcal{R}$ is the rectangle $[0, 5] \times [0, 1]$, and where h is the function

$$h : \mathbb{R}^2 \to \mathbb{R}, \qquad h(x, y) = e^{2x} + e^x e^y.$$

(2) *(Warm-up)*

(a) Consider the function

$$V : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}, \qquad V(x, y) = \ln(x^2 + y^2).$$

(i) Compute the gradient $\nabla V(x, y)$ for each $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(ii) Find $\nabla V(3, 4)$ and $\nabla V(-5, 12)$.

(b) Consider the function

$$w : \mathbb{R}^3 \to \mathbb{R}, \qquad w(x, y, z) = xy + xz + yz.$$

(i) Compute the gradient $\nabla w(x, y, z)$ for each $(x, y, z) \in \mathbb{R}^3$.

(ii) Find $\nabla w(-1, 1, 6)$.

(3) *(Warm-up)* Are the following parametric curves regular?

(a) *Quartic function:*

$$\mathbf{a} : \mathbb{R} \to \mathbb{R}^3, \qquad \mathbf{a}(t) = (t, 0, t^4).$$

(b) *No idea what to call this thing:*

$$\mathbf{b} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{b}(t) = ((t-1)^3, e^{(t-1)^2}).$$

(c) *Lemniscate of Gerono:*

$$\mathbf{c} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{c}(t) = (\cos t, \sin t \cos t).$$

(4) [**Marked**] Let g be the function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = \frac{2}{625} e^{\frac{x+y}{5}} (x + y),$$

and let C denote the region

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid -(5 - z) \leq x \leq (5 - z),\ -(5 - z) \leq y \leq (5 - z),\ 0 \leq z \leq 5\}.$$

(a) Sketch the region C.

(b) Compute the triple integral

$$\iiint_C f \, dV.$$

(5) [**Tutorial**] Answer the following:

(a) Let f be the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2 y,$$

and let D denote the triangular region

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1, \, |x| \le y\}.$$

(i) Sketch the region D on a Cartesian plane.

(ii) Compute the double integral

$$\iint_D f \, dA.$$

(b) Let Q denote the region

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le y + z, \, 0 \le y \le 1, \, 0 \le z \le 1\}.$$

(i) Sketch the region Q (or at least, do the best you can).

(ii) Use a triple integral to compute the volume of Q.

(6) *(Fun with cycloids)* Consider the parametric curve

$$\mathbf{c} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{c}(t) = (t - \sin t, 1 - \cos t).$$

(The path mapped out by $\mathbf{c}$ is known as a *cycloid.*)

(a) Show that $\mathbf{c}$ is not regular. At which $t \in \mathbb{R}$ do the values $|\mathbf{c}'(t)|$ vanish?

(b) Plot the image of $\mathbf{c}$ using a computer *(see the links on the QMPlus page)*. What happens at the points $\mathbf{c}(t)$ along the plot at which $|\mathbf{c}'(t)| = 0$?

(7) *(More parametric curves)* For each of the following parametric curves γ: (i) sketch, with the help of a computer, the image of γ, and (ii) determine whether γ is regular.

(a) *Cissoid of Diocles:*

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \left(\frac{t^2}{1 + t^2}, \frac{t^3}{1 + t^2} \right).$$

(b) *Witch of Agnesi:*

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \left(t, \frac{1}{1 + t^2} \right).$$

(c) *Tricuspoid:*

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (2\cos t + \cos(2t),\ 2\sin t - \sin(2t)).$$

(8) *(Reparametrise my hyperbola!)* Consider the following parametric curves:

$$\mathbf{a} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{a}(t) = (\cosh t, \sinh t),$$
$$\mathbf{b} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{b}(t) = \left(\sqrt{1+t^2}, t\right).$$

(a) Sketch the image of **b**.

(b) Show that both **a** and **b** are regular.

(c) Show that $\mathbf{a}(t) = \mathbf{b}(\sinh t)$ for any $t \in \mathbb{R}$. According to definition, what else must you to show in order to demonstrate that **a** and **b** are reparametrisations of each other?

(d) Finish what you started in (c)—show that **a** and **b** are reparametrisations of each other. *(You will not need advanced knowledge, but you will have to be extra resourceful.)*

(9) *(Numbers, Sets, and Functions revisited)* Let $\mathcal{P}$ denote the set of all regular parametric curves in $\mathbb{R}^n$. Given any two $\gamma_1, \gamma_2 \in \mathcal{P}$, we write $\gamma_1 \sim \gamma_2$ iff γ_1 is a reparametrisation of γ_2. Show that this $\sim$ defines an *equivalence relation* on $\mathcal{P}$.