

Typed-up notes for Week 2

& Week 3

have been uploaded onto

the Week 2 tab!

I'll edit them a bit more

next week though.

Office hours Friday 11-12

MB-B21

Wednesdays 1-2

Learning Cafe

Last Monday

Equivalence relation \mathcal{Q}
on a set S .

We write

$$[a] = [a]_{\mathcal{Q}}$$

for $a \in S$

to mean

$$\{b \in S \mid a \mathcal{Q} b\}$$

In particular, if \mathcal{Q} is an equivalence relation.

$$\text{from } [a] = [b]$$



existence for any $b \in \mathbb{P}^n$

class

$$a \mathcal{R} b.$$

represented by a .

(6)
[key issue:

n, m positive

integers,

Q

$$n^2 \mid m^2 \Rightarrow n \mid m?$$

To prove this we use

The fundamental theorem of

arithmetic

$$N = \prod_p p^{r_p}$$

product

$$r_p \geq 0$$

$$\forall p$$

$$= p_1^{r_1} \cdot p_2^{r_2} \cdot \dots$$

p_i 's are distinct
primes.

Def S a set.

A partition \mathcal{P} of S

is a set of subsets of S

satisfying the conditions

- $\emptyset \notin \mathcal{P}$

- If $A \& B \in \mathcal{P}$

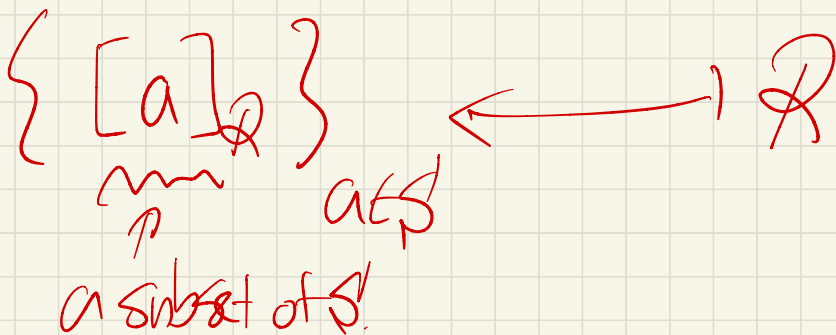
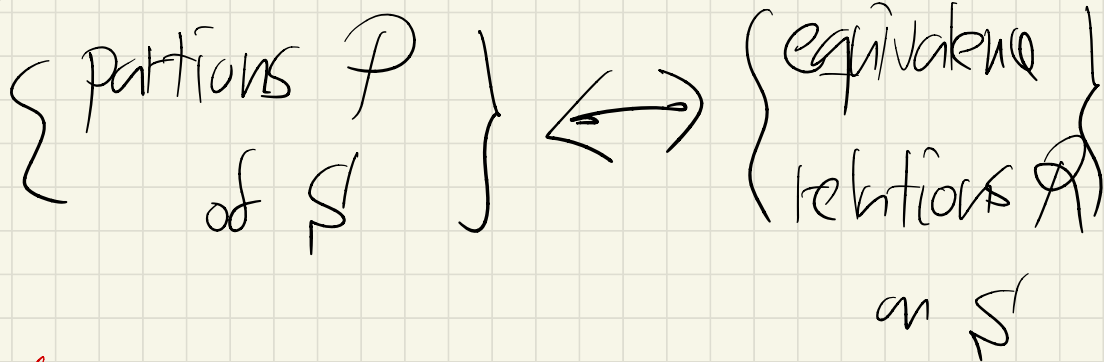
i.e. $A \& B$ are subsets
of S in \mathcal{P}

$\&$ distinct,

$$\text{then } A \cap B = \emptyset$$

- The union of all $A \in \mathcal{P}$ is S .

Theorem 9



$\mathcal{P} \mapsto$ Define $a \mathcal{Q} b$

if $a \mathcal{Q} b$ lies

in the same

$A \in \mathcal{P}$

Let's prove Theorem 9!

Suppose we are given an equiv.
relation \mathcal{Q} on S .

We need to check that

$$\{ [a]_{\mathcal{R}} \}$$

satisfies the conditions.

for partition.

- No element of $\{ [a]_{\mathcal{R}} \}$ is \emptyset .

Since \mathcal{R} is reflexive,

$$a \in [a]_{\mathcal{R}}$$

Therefore $[a]$ is non-empty.

• If $[a]$ and $[b]$ are
distinct,

$$\text{then } [a] \cap [b] = \emptyset$$

Equivalently,

$$\text{if } [a] \cap [b] \neq \emptyset,$$

$$\text{then } [a] = [b].$$

By assumption,

$$\exists c \in [a] \cap [b] \\ \neq \emptyset$$

$$\Rightarrow c \in [a]$$

$$c \in [b]$$

$$\Rightarrow a \mathcal{Q} c$$

i.e.

$$b \mathcal{Q} c$$

...

hold.

That means

$c \mathcal{Q} b$ also holds

(because \mathcal{Q} is symmetric)

Since \mathcal{Q} is transitive

...

...

\Rightarrow

$a \mathcal{Q} b$

$$\Rightarrow \underline{\underline{[a] = [b]}}$$

• The union T of $[a]$'s
 $a \in S$

equals S .

We need to check

$$\textcircled{1} \quad S \subseteq T$$

$$\textcircled{2} \quad T \subseteq S.$$

① By definition

$$\forall a \in S, [a] \subseteq S'$$

$$\begin{aligned} & \parallel \\ & \{ b \in S' \mid a \not\sim b \} \end{aligned}$$

Therefore

$$T \subseteq S'$$

(I've used the fact that

$$U \subseteq S'$$

$$V \subseteq S'$$

$$\text{then } U \cup V \subseteq S')$$

② $S \subseteq T$?

Let a be an element
in S .

(GOAL: $a \in T$)

The equivalence

$$[a] \subseteq T$$

OTOH, $a \in [a]$

Therefore, $a \in T$. \square

We've just proved that

equiv. relation $\mathcal{R} \implies$ partition \mathcal{P} .

Conversely, suppose we are given

a partition \mathcal{P} ,

we need to check that

the relation \mathcal{R}

(defined s.t. $a \mathcal{R} b$

$\iff a \& b$

lies in a part
in \mathcal{P} .)

satisfies the conditions.

(Exercise! Do it!)

Also check that

\mathcal{Q} so defined is

unique.

3.2 Congruences mod n .

Let n be a positive integer.

Def We define a relation

\equiv

on $S = \mathbb{Z}$ (the integers)

by: $a, b \in \mathbb{Z}$

$$a \equiv b \pmod{n}$$

(a is congruent
to $b \pmod{n}$)

if and only if n divides

$$b - a$$

$$\text{or } a - b$$

$P_{\text{top } 10}$ \equiv is an equivalence
relation
(congruence mod n
relation) on \mathbb{Z} .

According to Theorem 9,

this defines a partition on \mathbb{Z} .

What is it?

$$\{ [0], [1], \dots, [n-1] \}$$

$$[n]$$

"

$$[2n]$$

⋮

$$[n+1]$$

"

$$[-n+1]$$

"

$$[-3n+1]$$

By Theorem 9,

we know that

every integer

belongs to

exactly one

[] above.