Mathematical Tools for Asset Management
MTH6113

Dr Melania Nica

Practice Session 2

1. In the last Budget the Chancellor had announced the launch of a new three-year "Investment Guaranteed Growth" bond which will pay 2.2% annual return on an investment up to £3,000. There is a penalty for cashing in early, which equals 90 days' interest on the amount cashed in. The bond is available from today to anyone aged 16 or over.

   At the same time the Atom Bank offers a two-year bond which pays a 2.1 % annual return on any amount invested with a penalty of only 30 days' interest on the amount cashed in. You can only buy the Atom Bank bonds through an app. As these bonds have different characteristics (in terms of return, maturity, penalty for cashing in early and availability over the internet or not) you need to analyse very carefully what is your preference for them. After a lot of introspection, you realise that your preference for these two bonds is described by the utility function $U(X,Y) = (X + 200)(Y + 100)$ where X is the amount invested in the government bond and Y the amount invested in the Atom Bank bond. Your budget for investing in these two bonds is £3,000. Both the bonds are selling at par, i.e. you can assume that for £1 face value you pay £1, hence $p_X = p_Y = £1$

a. List the assumptions you would need to make about your preferences so that you can apply utility theory to find your optimum investment in each bond.

<u>Answer:</u>

Completeness, Transitivity, Local non-satiation or (More is better)

b. Explain what you understand by your indifference curve for these investments.

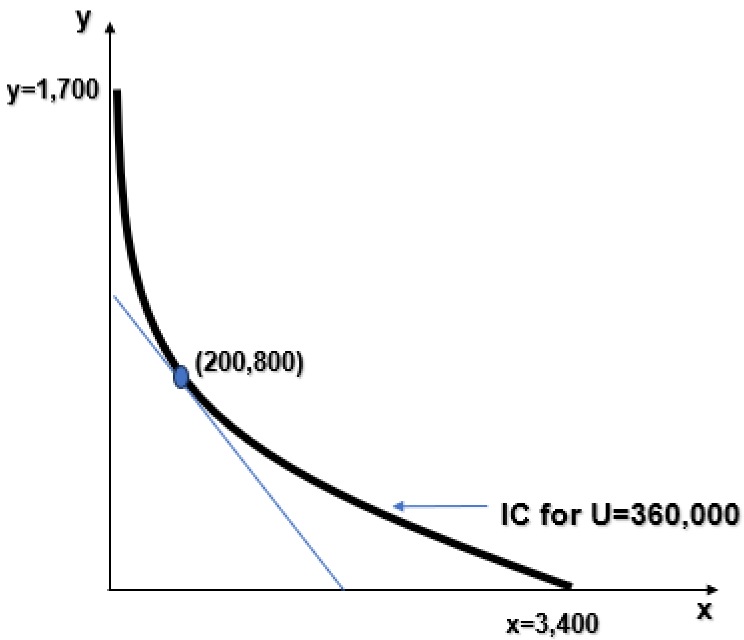
Answer: Indifference curve: a set of bundles (amounts invested in each bond) between which you are indifferent.

c. Write an equation for your indifference curve that goes through the point (X,Y) = (200 ,800). Sketch this indifference curve.

Answer:

$$U(X,Y) = (X + 200)(Y + 100) = 360{,}000$$

$$Y = \frac{360{,}000}{X + 200} - 100 = \frac{360{,}000 - 100X - 20{,}000}{X + 200} = \frac{340{,}000 - 100X}{X + 200}$$



d. What is your budget constraint and your marginal rate of substitution?

<u>Answer</u>

Budget constraint: $p_X X + p_Y Y \leq M$ or substituting the prices and available money to spend: $X + Y \leq 3{,}000$.

$$MRS = \frac{\partial U / \partial X}{\partial U / \partial Y} = \frac{Y + 100}{X + 200}$$

e. Write your optimization problem and find the optimum amounts invested in the two bonds.

<u>Answer</u>

Decision problem:

$max_{X,Y \geq 0}(X + 200)(Y + 100)$ such that

$X + Y \leq 3{,}000$

Two methods to solve for the solutions:

Either diagrammatically where slope of the budget line needs to be equal to the slope of the indifference curve or MRS:

$$MRS = \frac{\partial U/\partial X}{\partial U/\partial Y} = \frac{Y+100}{X+200}$$

Slope of budget line=1, hence at the optimum:

$\frac{Y+100}{X+200} = 1$ or $X + 200 = Y + 100$ (eq. 1)

Given that you will always use up all your available funds for higher utility:

$X + Y = 3{,}000$ (eq. 2)

Substituting eq. 1 into eq.2 you get:

$$X + X + 100 = 3{,}000$$
$$X^* = £1{,}450$$

and

$$Y^* = £1{,}550$$

The second method: maximize the Lagrangian function:

$$max_{X,Y\geq 0}\, L(X, Y, \lambda) = (X + 200)(Y + 100) + \lambda(3{,}000 - X - Y)$$

With the same solutions as before.

Make sure you verify the second order conditions:

Determinant of Bordered Hessian is positive

f. If these bonds had been offered in a different country by both a government and a private bank with histories of defaulting on debts would you still use the same method to find your optimal choice for investment? If yes, explain briefly why. If not, explain briefly what method would you use instead? Assume that now you know the expected returns of these bonds and their risks expressed as their standard deviation.

<u>Answer</u>

No, you cannot apply utility theory as the returns on the bonds are not certain anymore.

As you don't know the pdf of returns you cannot calculate the expected utility either (which you could max otherwise). We will apply Mean Variance Portfolio Theory which we will learn in Week 5.

2. A consumer has utility from consuming goods x_1 and x_2 $f(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$.

(a) If the price of x_1 is p_1 and the price of x_2 is p_2 while her income is m find this consumer's demand functions (optimal choice)

$$MRS = \frac{\partial U / \partial x_1}{\partial U / \partial x_2} = \frac{0.5 x_1^{-0.5}}{0.5 x_2^{-0.5}} = \frac{\sqrt{x_2}}{\sqrt{x_1}}$$

At the optimum:

$MRS = \dfrac{p_1}{p_2}$ and hence $p_1 \sqrt{x_1} = p_2 \sqrt{x_2} \Leftrightarrow x_1 = \dfrac{p_2^{\,2} x_2}{p_1^{\,2}}$

Substituting this expression in the budget constraint we get:

$$\frac{{p_2}^2 x_2}{p_1} + p_2 x_2 = m \Leftrightarrow \frac{x_2\left({p_2}^2 + p_1 p_2\right)}{p_1} = m \Leftrightarrow {x_2}^* = \frac{mp_1}{{p_2}^2 + p_1 p_2}$$

Similarly:

$${x_1}^* = \frac{mp_2}{{p_1}^2 + p_1 p_2}$$

(b) If $p_1 = £2$ is $p_2 = £1$ and the consumer has an income of £100 what are the demands for x_1 and x_2 of this consumer.

$$x_1^* = \frac{100}{2^2 + 2\times 1} = \frac{100}{6} = 16.67$$

$$x_2^* = \frac{100\times 2}{1^2 + 2\times 1} = \frac{200}{3} = 66.67$$

3. An investor, who prefers more to less, has preferences can be modelled by the utility function such that

$$U'(w) = 7.5 - 2w \ (w > 0)$$

a) Over what range can this utility function be satisfactorily applied?

The investor prefers more to less or:

$$U'(w) > 0 \Rightarrow 7.5 - 2w > 0 \Rightarrow 0 < w < 3.75$$

Show how the investor's absolute amount and relative amount of holdings in risky assets change as his wealth decreases.

$$U'(w) > 0 \Rightarrow 7.5 - 2w > 0 \Rightarrow 0 < w < 3.75$$

$$U''(w) = -2$$

b) Show how the investor's absolute amount and relative amount of holdings in risky assets change as his wealth increases.

Consider Arrow-Pratt measures:

$$A(w) = \frac{-U''(w)}{U'(w)} = \frac{2}{7.5-2w}$$

$$A'(w) = \frac{4}{(7.5-2w)^2} > 0$$ (noting denominator cannot be zero)

$$R(w) = \frac{-wU''(w)}{U'(w)} = \frac{2w}{7.5-2w}$$

$$R'(w) = \frac{2(7.5-2w)+4w}{(7.5-2w)^2} = \frac{15}{(7.5-2w)^2} > 0$$

Hence there is increasing ARA and RRA

4. A homeowner has initial wealth of £100,000 and utility function of wealth $U(w) = ln(w)$.

In the next year there is a chance that her house will suffer damage as follows:

Value of damage	Probability
£0	0.89
£5,000	0.1
£50,000	0.01

An insurer offers a policy that will fully protect the householder against losses arising from such damage in the next year. The insurer calculates that the premium charged for the policy should be 20% higher than the expected costs of claims.

a) Calculate the premium that the insurer will charge.

b) Calculate the maximum price that the householder is prepared to pay and hence determine whether the policy will be purchased.

a) Calculate the premium that the insurer will charge.

Expected cost of claims $= 0.89 \times 0 + 0.1 \times 5{,}000 + 0.01 \times 50{,}000 = £1{,}000$

So insurer will charge $1.2 \times 1{,}000 = £1{,}200$

b) Calculate the maximum price that the householder is prepared to pay and hence determine whether the policy will be purchased.

Maximum price householder will pay is c_x such that:

$U(w_0 - c_x) = E[U(w_0 + X)]$ where w_0 is the initial wealth of 100,000 and X is the random loss.

$$ln(100{,}000 - c_x) = 0.89 \times ln(100{,}000) + 0.1 \times ln(95{,}000) + 0.01 \times ln(50{,}000)$$
$$= 11.50086$$

Thus $c_x = -exp(11.50086) + 100{,}000 = 1{,}198.84$

Hence householder will not purchase the policy.