MTH6127 **Metric Spaces and Topology**

Course work 1 **26 January 2024**

1. For each of the following subsets of $\mathbb{R}$, determine the supremum and decide whether it is a maximum. Justify your answers.

   (a) $A = \{x : x^2 - 2 < 0\}$,
   One has $A = (-\sqrt{2}, \sqrt{2})$ and hence $\sup A = \sqrt{2}$, the maximum does not exist.

   (b) $B = \{x^2 - 2 : -2 \leq x < 2\}$,
   One has $B = [-2, 2]$ and therefore $\sup B = 2$ and $\max B = 2$.

   (c) $C = \{1 - 1/n^2 : n = 1, 2, 3, \ldots\}$
   $\sup C = 1$, the maximum does not exist.

   (d) $D = \{1 + 1/n^3 : n = 1, 2, 3, \ldots\}$.
   The sequence is decreasing and therefore $\sup D = 2$ and $\max D = 2$ is achieved for $n = 1$.

2. Let $X = \{0, 1\}^\omega$ be the set of all infinite sequences formed of 0s and 1s. For $x, y \in X$, define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|,$$

   where $x = (x_n)$ and $y = (y_n)$. Prove that d is well-defined and is a metric on X.

   The series defining d converges since $|x_n - y_n| \leq 1$ and $\sum_{n=1}^{\infty} 2^{-n} = 1$.

   If $x, y, z \in X$, where $x = (x_n)$, $y = (y_n)$, $z = (z_n)$, then for any n

$$|x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|$$

   implying

$$\sum 2^{-n}|x_n - y_n| \leq \sum 2^{-n}|x_n - z_n| + \sum 2^{-n}|z_n - y_n|,$$

   i.e. $d(x, y) \leq d(x, z) + d(z, y)$. M1 and M2 are obvious.

3. Prove that if (X, d) is a metric space then $\sigma : X \times X \to \mathbb{R}$ is a metric where

$$\sigma(x, y) = \min\{d(x, y), 1\}.$$

   Given $x, y, z \in X$, denote $d(x, z) = a$, $d(x, y) = b$, $d(y, z) = c$ and $\sigma(x, z) = a'$, $\sigma(x, y) = b'$, $\sigma(y, z) = c'$ We may assume that $a \geq b \geq c$ and $a \leq b + c$.

   If $a \leq 1$ then $b \leq 1$ and $c \leq 1$, and $a = a'$, $b = b'$, $c = c'$ and $a' \leq b' + c'$.

   If $a \geq 1 \geq b$ then $a' < a \leq b' + c'$ since in this case $b = b'$ and $c = c'$.

   If $b \geq 1$ then $a' \leq b' + c'$ as in this case $a' = 1$ and $b' = 1$.

4. A metric space X is said to be bounded if there is some number $M > 0$ such that

$$d(x, y) \leq M$$

for any $x, y \in X$. Show that for any metric space (X, d), the metric space (X, σ) (as defined in the previous question) is bounded. Show also that the metric of example 2 is bounded.

It is clear that $\sigma(x, y) \leq 1$, i.e. the metric σ is bounded. The metric of example 2 also satisfies $d(x, y) \leq \sum_{n=1}^{\infty} 2^{-n} = 1$.

5. The Euclidean norm of a vector $p = (p_1, p_2) \in \mathbb{R}^2$ is defined as $\|p\|_2 = \sqrt{p_1^2 + p_2^2}$. For $p, q \in \mathbb{R}^2$ define $d(p, q)$ by

$$d(p, q) = \begin{cases} 0, & \text{if } p = q; \\ \|p\|_2 + \|q\|_2, & \text{otherwise.} \end{cases}$$

Prove that d is a metric on $\mathbb{R}^2$.

We want to show that for a triple of points $p, q, r \in \mathbb{R}^2$ one has $d(p, q) \leq d(p, r) + d(r, q)$. We may assume that $r \neq p$ and $r \neq q$ since otherwise our statement is obvious. Besides, for the same reason, we may assume that $p \neq q$. Then our statement reduces to $||p|| + ||q|| \leq ||p|| + ||r|| + ||r|| + ||q||$ which is clearly satisfied since $||r|| \geq 0$.