# Recap Quiz (3 min)

Consider the following LP.

objective function

$$\begin{aligned} \text{maximise} \quad & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 = 3 \\ & x_1 + 3x_2 \leq 6 \\ & x_1 \geq 0, \; x_2 \leq 0, \; x_3 \text{ unrestricted.} \end{aligned}$$

goal

constraints

sign restrictions

1) Label the
(a) goal
(b) objective function
(c) constraints
(d) sign restrictions

2) If we transform this LP into standard inequality form
(a) how many variables will it have? 4
(b) how many constraints will it have? 3

3) Write down (in terms of matrices) what a general linear program in standard inequality form looks like.

$$\begin{aligned} \max \quad & \underline{c}^T \underline{x} \\ \text{sub to} \quad & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

$A$ matrix
$\underline{c}, \underline{b}$ vectors
$\underline{x}$ vector of variables

**Defn** Consider an arbitrary LP (linear programme) with $n$ variables $x_1, x_2, \dots, x_n$. Write $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

A feasible solution to the LP is an assignment of real values to the variables that satisfy all sign restrictions and all constraints.

An optimal solution to the LP is a feasible solution that achieves the goal

So for a LP in standard inequality form
$$\begin{aligned} \max\ & \underline{c}^T \underline{x} \\ \text{sub to}\ & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

a feasible solution is any vector $\underline{x}$ that satisfies $A\underline{x} \leq \underline{b}$ and $\underline{x} \geq \underline{0}$

$\underline{x}$ is an optimal solution if $\underline{x}$ is feasible and

$$\underline{c}^T \underline{x} \geq \underline{c}^T \underline{x}'$$

for any other feasible solution $\underline{x}'$

## Task

$$\begin{aligned} \text{maximise}\ & x_1 + x_2 \\ \text{subject to}\ & -x_1 + 2x_2 \leq 3 \\ & -2x_1 + x_2 \geq -6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Which are feasible solutions?
Which three are not optimal solutions?

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

# Task

maximise $x_1 + x_2$

subject to
$$-x_1 + 2x_2 \leqslant 3$$
$$-2x_1 + x_2 \geqslant -6$$
$$x_1, x_2 \geqslant 0$$

Which are feasible solutions?

Which three are <u>not</u> optimal solutions?

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

| | $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 100 \end{pmatrix}$ | $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ |
|---|---|---|---|---|
| feasible | ✗ sign restrictions violated | ✓ | ✗ first constraint violated | ✓ |
| optimal | ✗ not feasible | ✗ $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ is better. | ✗ not feasible | ? |

Gapped notes available before lecture on @Mplus

Look again at week 1 seminar questions 3, 4 now that you have definitions.

**Example 1.1.** A university student is planning her daily food budget. Based on the British Nutrition Foundation's guidelines for an average female of her age she should consume the following daily amounts of vitamins:

| | Vitamin | mg/day |
|---|---|---|
| B1 | Thiamin | 0.8 |
| B2 | Riboflavin | 1.1 |
| B3 | Niacin | 13 |
| | Vitamin C | 35 |

After doing some research, she finds the following cost, calories, and vitamins (in mg) per serving of several basic foods:

| Food | Cost | Thiamin | Riboflavin | Niacin | Vitamin C |
|---|---|---|---|---|---|
| Bread | £0.25 | 0.1 | 0.1 | 1.3 | 0.0 |
| Beans | £0.60 | 0.2 | 0.1 | 1.1 | 0.0 |
| Cheese | £0.85 | 0.0 | 0.5 | 0.1 | 0.0 |
| Eggs | £1.00 | 0.2 | 1.2 | 0.2 | 0.0 |
| Oranges | £0.80 | 0.2 | 0.1 | 0.5 | 95.8 |
| Potatoes | £0.50 | 0.2 | 0.1 | 4.2 | 28.7 |

How can the student meet her daily requirements as cheaply as possible?

# Mathematical Program

Goal

Objective Function

$$
\begin{array}{rl}
\text{minimise} & 0.25x_1 + 0.60x_2 + 0.85x_3 + 1.00x_4 + 0.80x_5 + 0.50x_6 \\
\text{subject to} & 0.1x_1 + 0.2x_2 + 0.0x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 \geq 0.8 \\
& 0.1x_1 + 0.1x_2 + 0.5x_3 + 1.2x_4 + 0.1x_5 + 0.1x_6 \geq 1.1 \\
& 1.3x_1 + 1.1x_2 + 0.1x_3 + 0.2x_4 + 0.5x_5 + 4.2x_6 \geq 13 \\
& 0.0x_1 + 0.0x_2 + 0.0x_3 + 0.0x_4 + 95.8x_5 + 28.7x_6 \geq 35 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}
$$

Constraints

Variables

Restrictions

## General approach to translating problem into LP

① Identify choices that have to be made and introduce decision variable for each such choice

② Introduce a dummy variable for all other relevant quantities

③ Express the goal/constraints in terms of the variables introduced, (decision variables)
Include sign restrictions
Dummy variables generally unrestricted.

④ Make substitutions to simplify LP (optional step).

**Example 2.1.** A factory makes 2 different parts (say, part $X$ and part $Y$). Their plant has 4 separate processes in place: there are two older processes (say, process 1 and 2) that produce parts $X$ and $Y$ directly, as well as two different integrated processes for producing both $X$ and $Y$ simultaneously. The 4 processes can be run simultaneously, but require labour, raw metal, and electricity. The hourly inputs and outputs for each process are as follows:

| | Outputs | | Inputs | | |
|---|---|---|---|---|---|
| Process | $X$ | $Y$ | Metal | Electricity | Labour |
| 1 | 4 | 0 | 100 kg | 800 kWh | 16 hrs |
| 2 | 0 | 1 | 70 kg | 600 kWh | 16 hrs |
| 3 | 3 | 1 | 120 kg | 2000 kWh | 50 hrs |
| 4 | 6 | 3 | 270 kg | 4000 kWh | 48 hrs |

~~In a typical day~~, the plant has an available stock of 6000 kg of metal, and the has budgeted 100000 kWh of power usage, 1000 hours of labour. Suppose that each part $X$ sells for £1000 and each part $Y$ sells for £1800. How should production be scheduled to maximise daily revenue?

**Task:** What choices have to be made ①?

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| 4 | 6 | 3 | 270 kg | 4000 kWh | 48 hrs |


Available

6000 kg metal
100000 kWh electric
1000 hours labour

X sells £1000/unit
Y £1800/unit

revenue

maximise $1000x + 1800y$

sub to
$$x = 4p_1 + 3p_3 + 6p_4$$
$$y = p_2 + p_3 + 3p_4$$
$$m = 100p_1 + 70p_2 + 120p_3 + 270p_4$$
$$e = 800p_1 + 600p_2 + 2000p_3 + 4000p_4$$
$$l = 16p_1 + 16p_2 + 50p_3 + 48p_4$$
$$m \le 6000$$
$$e \le 100000$$
$$l \le 1000$$
$p_1, p_2, p_3, p_4 \geq 0$ $\quad x, y, m, e, l$ unrestricted

Variables

decision
$p_1$ = # hours of process 1
$p_2$ 2
$p_3$ 3
$p_4$ 4

$x$ = units of X produced
$y$ = units of Y produced
$m$ = kg metal needed
$e$ = kWh electric needed
$l$ = hours of labour needed

dummy

<u>Rem</u> Can simplify LP by eliminating non-decision variables

maximise $1000(4P_1 + 3P_3 + 6P_4) + 1800(P_2 + P_3 + 3P_4)$

sub to $\cancel{x = 4P_1 + 3P_3 + 6P_4}$

$\cancel{y = P_2 + P_3 + 3P_4}$

$\cancel{m =}\ 100P_1 + 70P_2 + 120P_3 + 270P_4 \leq 6000$

$\cancel{e =}\ 800P_1 + 600P_2 + 2000P_3 + 4000P_4 \leq 100000$

$\cancel{l =}\ 16P_1 + 16P_2 + 50P_3 + 48P_4 \leq 1000$

$\cancel{m \leq 6000}$

$\cancel{e \leq 100000}$

$\cancel{l \leq 1000}$

$P_1, P_2, P_3, P_4 \geq 0$ $\cancel{x, y, m, e, l \text{ unrestricted}}$

Both LPs are correct answers to question

Careful about making substitutions if variables are restricted.

**Example 2.2.** Suppose that our factory in Example 2.1 wants to determine its daily operating budget. It has determined that there is daily demand for 120 parts $X$ and 50 parts $Y$. Suppose now that there is an unlimited amount of metal, electricity, and labour available, but the cost of metal is £5 per kg, the cost of electricity is £0.15 per kWh, and the cost of labour is £20 per hour. How can it schedule production to meet its demand as cheaply as possible?

| | Outputs | | Inputs | | |
|---|---|---|---|---|---|
| Process | $X$ | $Y$ | Metal | Electricity | Labour |
| 1 | 4 | 0 | 100 kg | 800 kWh | 16 hrs |
| 2 | 0 | 1 | 70 kg | 600 kWh | 16 hrs |
| 3 | 3 | 1 | 120 kg | 2000 kWh | 50 hrs |
| 4 | 6 | 3 | 270 kg | 4000 kWh | 48 hrs |

Demand 120 parts X
50 parts Y

metal costs £5/kg
electric £0.15/kWh
labour £20/hour

minimise $5m + 0.15e + 20l$

sub to $x = 4P_1 + 3P_3 + 6P_4$
$y = P_2 + P_3 + 3P_4$

$m = 100P_1 + 70P_2 + 120P_3 + 270P_4$
$e = 800P_1 + 600P_2 + 2000P_3 + 4000P_4$
$l = 16P_1 + 16P_2 + 50P_3 + 48P_4$

$x \geq 120$
$y \geq 50$

$P_1, P_2, P_3, P_4 \geq 0$ $x, y, m, e, l$ unrestricted.

variables

decision $\left\{ \begin{array}{ll} P_1 = \text{\# hours of process} & 1 \\ P_2 & 2 \\ P_3 & 3 \\ P_4 & 4 \end{array} \right.$

$\left\{ \begin{array}{l} x = \text{\# parts X produced} \\ y = \text{\# parts Y produced} \\ m = \text{kg metal used} \\ e = \text{kWh electric used} \\ l = \text{hours of labour used} \end{array} \right.$
dummy

Could simplify this if we want.
Also acceptable to have $x = 120$, $y = 50$ instead of $x \geq 120$, $y \geq 50$

**Example 2.3.** Suppose that our factory in the previous 2 examples now wants to find a production schedule that maximises its daily *profits* defined as revenue minus costs. How can this be done? You should assume that any amount of resources are available, and that any number of parts can be sold (where the prices are given as in the previous 2 examples).

| | Outputs | | Inputs | | |
|---|---|---|---|---|---|
| Process | $X$ | $Y$ | Metal | Electricity | Labour |
| 1 | 4 | 0 | 100 kg | 800 kWh | 16 hrs |
| 2 | 0 | 1 | 70 kg | 600 kWh | 16 hrs |
| 3 | 3 | 1 | 120 kg | 2000 kWh | 50 hrs |
| 4 | 6 | 3 | 270 kg | 4000 kWh | 48 hrs |

X sells at £1000 /unit
Y sells at £1800 /unit.

metal costs £5 per kg
electric £0.15 per kWh
labour £20 per hour

Variables
$\underbrace{P_1, \ldots, P_4}_{\text{decision}}, \underbrace{x, y, m, e, l}_{\text{dummy}}$

maximise revenue − cost

$$= (1000x + 1800y) - (5m + 0.15e + 20l)$$

subject to

$$x = 4P_1 + 3P_3 + 6P_4$$
$$y = P_2 + P_3 + 3P_4$$
$$m = 100P_1 + 70P_2 + 120P_3 + 270P_4$$
$$e = 800P_1 + 600P_2 + 2000P_3 + 4000P_4$$
$$l = 16P_1 + 16P_2 + 50P_3 + 48P_4$$
$$P_1, P_2, P_3, P_4 \geq 0 \qquad x, y, m, e, l \text{ unrestricted.}$$

**Example 2.4.** A medical testing company is making diagnostic tests. Each test requires a combination of 3 different reagents:

| Test | Reagents Needed | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| Standard | 0.9 ml | 1.2 ml | |
| Rapid | 1.5 ml | | 1.0 ml |

Each reagent can be synthesised from a combination of more basic chemicals (let's call them chemical A, B, and C), which requires some amount of laboratory time. Additionally, these reagents can be purchased from a supplier for a listed price, and any extra reagent that the company produces can also be sold to the supplier for this price. The relevant materials and costs are summarised in the following table:

| Reagent | Chemicals Needed | | | Lab time to synthesise | Price |
|---|---|---|---|---|---|
| | A | B | C | | |
| 1 | 1.0 ml | 0.3 ml | 1.5 ml | 0.02 hrs/ml | £2.40/ml |
| 2 | 0.5 ml | 0.2 ml | 1.0 ml | 0.04 hrs/ml | £1.60/ml |
| 3 | 0.2 ml | 1.8 ml | 0.6 ml | 0.05 hrs/ml | £1.50/ml |

The company has taken on a contract to produce 1000 standard tests and 2300 rapid tests. It has 100 hours of laboratory time available at a cost of £150 per hour, 1100ml of chemical A, 1250ml of chemical B, and 1800ml of chemical C available. Additionally, it can purchase and sell an unlimited amount of each reagent for the specified price. Find a production plan that fulfils the contract at the lowest net cost, taking into account any money recovered by the sale of excess reagents.

chemicals/
lab time

↓

Reagents ⟷ buy/sell

↓

Tests

Must produce
1000 standard tests
2300 rapid tests

Available
1100 ml chem A
1250 ml B
1800 C
100 hours lab time
lab time costs £150/hour

Reagent 1 needed $= 1000 \times 0.9 + 2300 \times 1.5 = 4350$ ml
2 $= 1000 \times 1.2 = 1200$ ml
3 $= 2300 \times 1.0 = 2300$ ml

| Test | Reagents Needed | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| Standard | 0.9 ml | 1.2 ml | |
| Rapid | 1.5 ml | | 1.0 ml |

| Reagent | Chemicals Needed | | | Lab time | Price |
|---|---|---|---|---|---|
| | A | B | C | to synthesise | |
| 1 | 1.0 ml | 0.3 ml | 1.5 ml | 0.02 hrs/ml | £2.40/ml |
| 2 | 0.5 ml | 0.2 ml | 1.0 ml | 0.04 hrs/ml | £1.60/ml |
| 3 | 0.2 ml | 1.8 ml | 0.6 ml | 0.05 hrs/ml | £1.50/ml |

Must produce
1000 standard tests
2300 rapid tests

Available
1100 ml chem A
1250 ml B
1800 C
100 hours lab time
lab time costs £150/hour

Reagent 1 needed $= 1000 \times 0.9 + 2300 \times 1.5 = 4350$ ml
2 $= 1000 \times 1.2 = 1200$ ml
3 $= 2300 \times 1.0 = 2300$ ml

minimise $c^*$
subject to

$c^* = 2.4 s_1 + 1.6 s_2 + 1.5 s_3 + 150 l$
$a = 1.0 r_1 + 0.5 r_2 + 0.2 r_3$
$b = 0.3 r_1 + 0.2 r_2 + 1.8 r_3$
$c = 1.5 r_1 + 1.0 r_2 + 0.6 r_3$
$l = 0.02 r_1 + 0.04 r_2 + 0.6 r_3$
$a \leq 1100$
$b \leq 1250$
$c \leq 1800$
$l \leq 1000$ ↙ 100 [mistake in lecture]
$r_1 + s_1 \geq 4350$
$r_2 + s_2 \geq 1200$
$r_3 + s_3 \geq 2300$

Variables
$c^*$ = cost
decision
$r_1$ = ml reagent 1 produced
$r_2$ = 2
$r_3$ = 3
$s_1$ = ml reagent 1 bought
$s_2$ 2
$s_3$ 3
$a$ = ml of chem A needed
$b$ B
$c$ C
$l$ = hours of lab time

$r_1, r_2, r_3 \geq 0$, $s_1, s_2, s_3$ unrestricted, $a, b, c, l, c^*$ unrestricted

# Transportation problem

**Example 2.5.** A mining company has 2 mines, where ore is extracted, and 3 warehouses, where ore is stored. Currently, there is 45Mg of ore divided amongst the mining locations. In order to prepare it for sale, this ore needs to be distributed to the warehouses. The amount of ore available at each mine, and the amount of ore required at each warehouse is as follows:

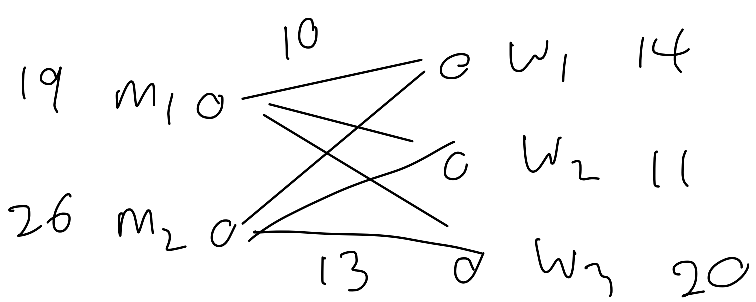
| | Ore Available |
|---|---|
| Mine 1 | 19 |
| Mine 2 | 26 |

| | Ore Required |
|---|---|
| Warehouse 1 | 14 |
| Warehouse 2 | 11 |
| Warehouse 3 | 20 |

Due to different distances and shipping methods, the cost (in thousands of pounds) to ship 1 Mg depends on where it is being shipped from and where it is being shipped to, as follows:

| | Warehouse 1 | Warehouse 2 | Warehouse 3 |
|---|---|---|---|
| Mine 1 | 10 | 5 | 12 |
| Mine 2 | 9 | 7 | 13 |

Suppose that these costs scale up linearly in the amount of ore that is shipped (for example, it costs $3 \cdot 10$ to ship 3Mg of ore from Mine 1 to Warehouse 1. How should we send the ore from the mines to the warehouses to minimise the overall transportation cost?




Variables

$x_{ij}$ = Mg of ore moved from mine $i$ to warehouse $j$

$i = 1, 2$

$j = 1, 2, 3$

minimise

$$10x_{11} + 5x_{12} + 12x_{13} + 9x_{21} + 7x_{22} + 13x_{23}$$

sub to

$$x_{11} + x_{12} + x_{13} \leq 19$$
$$x_{21} + x_{22} + x_{23} \leq 26$$
$$x_{11} + x_{21} \geq 14$$
$$x_{12} + x_{22} \geq 11$$
$$x_{13} + x_{23} \geq 20$$
$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

e.g. can write

$$\sum_{j=1}^{3} x_{2j} \leq 26$$

Transportation problems    Typically

- Move resources from some sources to some destinations
- Different cost of moving resource from source i to destination j

  Want to minimise total cost.

- some amount of resource available at each source
  (get one constraint for each source)
- some target amount of resource needed at each destination
  (get one constraint for each destination).

Ask following as question

An ebay trader has a £100 available and there are 3 types of items on sale today (with unlimited availability)
Tomorrow when the sale ends the trader will sell the items at a higher price.

The trader has £100.

| Item | Buying Price | Selling price |
|---|---|---|
| A | 90 | 180 |
| B | 20 | 38 |
| C | 60 | 117 |

Which items should he buy today to maximise profit.

a = number of item A bought
b = B
c = C

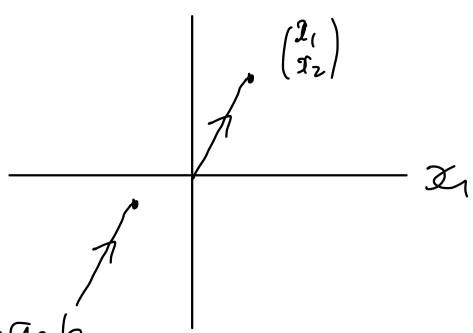
maximise $90a + 18b + 57c$
subject $90a + 20b + 60c \leq 100$
$a, b, c \geq 0$

optimal solution $a = \frac{10}{9}$, $b = 0$, $c = 0$ profit = £100
but want integers

optimal integer solution $a = 0$, $b = 2$, $c = 1$ profit = £93.

Facts about geometry of $\mathbb{R}^2$

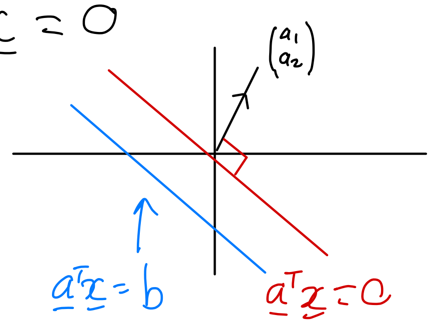
$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ can be thought of as a point in $\mathbb{R}^2$ or as a vector




If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ $\underline{a}^T \underline{b} = a_1 b_1 + a_2 b_2$

If $\underline{a}^T \underline{b} = 0$ means $\underline{a}$ and $\underline{b}$ are perpendicular.

Fix $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. Which $\underline{x} \in \mathbb{R}^2$ satisfy $\underline{a}^T \underline{x} = 0$




ans: all $\underline{x}$ on the line perpendicular to $\underline{a}$ that goes through $\underline{0}$.

$$a_1 x_1 + a_2 x_2 = 0$$

Which $\underline{x} \in \mathbb{R}^2$ satisfy $\underline{a}^T \underline{x} = b$ $b \in \mathbb{R}$

ans: all $\underline{x}$ on the line perpendicular to $\underline{a}$ but through some other point e.g. $b/a_2$.

$$a_1 x_1 + a_2 x_2 = b \quad \text{goes through} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ b/a_2 \end{pmatrix}$$

What happens to the line $\underline{a}^T \underline{x} = b$ as $b$ increases.

ans: it stays perpendicular to $\underline{a}$ but moves in the direction of $\underline{a}$.