Week 2

2.2 Basic structure of One-period Binomial model

Def 2.1 One-period Binomial Model

Consider a model of asset prices $S(i)$, where $i = 0, 1$,

$S(0) \triangleq S$

$S(1) = Su$ or Sd

u, d positive real numbers $0 < d < u$

r: nominal interest rate per time period

Q: Under what condition on u, d, r, there is no arbitrage.

$$
\begin{tikzcd}[row sep=small]
0 & 1 \\
 & S(0)\cdot u \\
S(0) \arrow[ur] \arrow[dr] & \\
 & S(0)\cdot d
\end{tikzcd}
$$

Theorem 2.1 (The Arbitrage Theorem)

Given a set of bets with returns $r_1, \cdots, r_n$,

1. There exists a prob vector $\vec{p} = (p_1, \cdots, p_m)$ s.t.

   $p_1 > 0, \cdots p_m > 0$

   For each $i = 1, \cdots, n$

   $p_1 r_i(1) + p_2 r_i(2) + \cdots + p_m r_i(m) = 0.$ (1)

2. There is an arbitrage.

A: 2 possible outcomes: $S(1) = Sd$ or $S(1) = Su$

Using The Arbitrage Theorem, Eq (1):

$\underline{p_1 r_1(1) + p_2 r_1(2)} = 0$ (2)

Return functions:

$$\begin{tikzcd} 0 \arrow[r, dash] & 1 \\ \frac{S(1)}{1+r} & S(1) \arrow[l, bend left] \end{tikzcd}$$

return at time $0 = \frac{S(1)}{1+r} - S$

$r_1(1) = \underbrace{\frac{Su}{1+r} - S}_{\text{PV of return if asset price moves up}}$ and $r_1(2) = \underbrace{\frac{Sd}{1+r} - S}_{\text{PV of return if asset price moves down}}$

$$\begin{cases} P_1\left(\frac{Su}{1+r} - S\right) + P_2\left(\frac{Sd}{1+r} - S\right) = 0 \\ P_2 = 1 - P_1 \end{cases}$$

$\uparrow$ price

$$\frac{P_1 u}{1+r} + \frac{(1-P_1)d}{1+r} - 1 = 0 \Rightarrow P_1 = \frac{1+r-d}{u-d}, \quad P_2 = \frac{u-1-r}{u-d}$$

$$P_1 \geq 0, \quad P_2 \geq 0, \quad \Rightarrow \quad 1+r \geq d, \quad u \geq 1+r \quad \Rightarrow \quad d \leq 1+r \leq u$$

P_1 and P_2 are call risk-neutral Prob (RNP) $\boxed{d < 1+r < u}$

Def 2.2 Risk Neutral Measure

~ is a prob measure s.t each price is exactly equal to the discounted expectation of the price under the measure.

Such measure exists iff the arbitrage-free.

Homework:

real-world prob measure, risk-neutral Prob measure

$\hat{p}_1$ $\downarrow$ hate risk — higher return to compensate for the risk

p_1 $\downarrow$ don't care about risk — E ✓ $\mathrm{Var}\,X$ ✗

$$\boxed{\hat{p}_1 > p_1}$$

$p_1 \uparrow$, $E \uparrow$ $\uparrow$ return

A to Q2 on slide 18

Hint: Eq (1) Arbitrage Theorem

Step 2: $r_2(1) = \underbrace{\frac{(Su-k)^+}{1+r} - c}_{\text{up}}$, $r_2(2) = \underbrace{\frac{(Sd-k)^+}{1+r} - c}_{\text{PV of return if asset price moves down}}$

r_2: return func at time 0

Step 3: Arbitrage Theorem

step 1: $0 \longrightarrow 1$

CF: $-C$ $\quad$ $\begin{cases} (Su-K)^+, & p_1 \\ (Sd-K)^+, & p_2 \end{cases}$

$$p_1 r_2(1) + p_2 r_2(2) = 0$$

$$\Rightarrow p_1 \left[\frac{(Su-K)^+}{1+r} - C \right] + p_2 \left[\frac{(Sd-K)^+}{1+r} - C \right] = 0$$

$$\Rightarrow C = p_1 \frac{(Su-K)^+}{1+r} + p_2 \frac{(Sd-K)^+}{1+r}$$

$$\boxed{(Sd-K)^+ = \max(Sd-K, 0)}$$

If: $Sd < K < Su$

$$\Rightarrow C = p_1 \frac{(Su-K)}{1+r} = \frac{(1+r-d)(Su-K)}{(1+r)(u-d)}$$

p_1 the same Q1

Q3. No-arbitrage price of the derivative defined on Slide 21

Step 2: $p_1 V_1 + p_2 V_2 - C(1+r) = 0$

$$C = \frac{1}{1+r} p_1 V_1 + \frac{1}{1+r} p_2 V_2 = \frac{1+r-d}{(1+r)(u-d)} V_1 + \frac{u-1-r}{(1+r)(u-d)} V_2$$

Q4: Slide ~~2~~ 25 + 26

Investment 1

0 —— 1

Invest in share $-x \cdot S$

Deposit in bank $-y$

$$\begin{cases} xSu & p_1 \\ xSd & p_2 \end{cases}$$

$y(1+r)$

Investment 2

0 —— 1

Invest in derivative

$-C$

$$\begin{cases} V_1 & p_1 \\ V_2 & p_2 \end{cases}$$

Investment 1 = Investment 2

$$\begin{cases} \text{up:} & xSu + y(1+r) = V_1 \\ \text{down:} & xSd + y(1+r) = V_2 \end{cases}$$

$$\Rightarrow \boxed{x = \frac{V_1 - V_2}{S(u-d)}}, \quad y = \frac{uV_2 - dV_1}{(1+r)(u-d)}$$

Step 1:

0 —— 1

Buy the derivative $-C$

$$\begin{cases} \text{Payoff} = V_1 & p_1 \\ \text{Payoff} = V_2 & p_2 \end{cases}$$

Borrow from Bank

cash: C

Return bank

$-C(1+r)$

0 $= 0$

W2 6

$$C = xS + y = \frac{V_1 - V_2}{S(u-d)} \cdot S + \frac{uV_2 - dV_1}{(1+r)(u-d)}$$

$$= \frac{1}{1+r}\left(\underbrace{\boxed{\frac{1+r-d}{u-d}}}_{p_1} V_1 + \underbrace{\boxed{\frac{u-1-r}{u-d}}}_{p_2} V_2 \right)$$

$$C = \frac{1}{1+r}\left(p_1 V_1 + p_2 V_2 \right)$$

The law of One price

3. Multiperiod Binomial model

Def 3.1 Multiperiod Binomial model

$S(t)$: Share price, where $t = 0, 1, \cdots, n$, s.t. $S(t-1)$

$$S(t) = \begin{cases} u\, S(t-1) \\ d\, S(t-1) \end{cases}$$

$S(0) = S$, $0 < d < u$. r: nominal interest rate per time period

$\vec{S} = (S(0), S(1), \cdots, S(n))$

$\vec{i} = (i_1, \cdots, i_n)$ s.t. for $1 \le j \le n$, $i_j = \begin{cases} 1 & \text{if } S(j) = uS(j-1) \\ 0 & \text{if } S(j) = dS(j-1) \end{cases}$

One-to-one correspondence between $\vec{S}$ and $\vec{i}$

For any time $k \in \{1, \cdots, n\}$, we have

$$S(k) = S\, u^{\sum_{j=1}^{k} i_j} d^{k - \sum_{j=1}^{k} i_j}$$

$$P(\vec{i}) \equiv P(i_1, \cdots, i_n) = p^{\sum_{j=1}^{n} i_j} (1-p)^{n - \sum_{j=1}^{n} i_j}, \quad \text{where } p = \frac{1+r-d}{u-d}$$

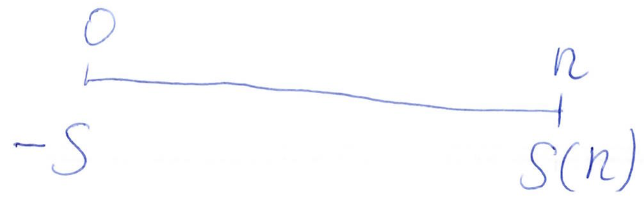
path: 1, 0, 0, 1 ⋯
u d d u ⋯

prob of up

$$P(X_1 = i_1, \cdots, X_n = i_n) = p^{\sum_{j=1}^{n} i_j} (1-p)^{n - \sum_{j=1}^{n} i_j}, \quad \text{where } p = \frac{1+r-d}{u-d}$$

$X_j \sim$ Bernoulli(p) iid

Example



I. Buy a share at time 0 for S

Sell it at time n for $S(n)$.

return func: $r(i_1, i_2, \cdots, i_n)$

According Theorem 3.1 (slide 37):

$$\sum_{i_1, i_2 \cdots i_n} r(i_1, i_2, \cdots, i_n) P(X_1 = i_1, \cdots, X_n = i_n)$$

$$= \tilde{E}[r(X_1, X_2, \cdots, X_n)] = 0$$ ← no arbitrage

$\tilde{E}$ is the expectation computed over the risk-neutral probability.

E real-world

II. $Y = \sum_{i=1}^{n} X_i$ ← total number of steps ... up

$Y \sim$ Binomial

$$P(Y=y) = \binom{n}{y} \underset{\substack{\uparrow \\ \text{Prob up}}}{p^y} \underset{\substack{\uparrow \\ \text{Prob down}}}{(1-p)}^{n-y}$$

n: total ~~pe~~ number of periods

y: total number of steps .. up

Q: no-arbitrage price C of a European call opition.

K, $T=n$

Theorem 3.2

$$C = \frac{1}{(1+r)^n} \sum_{y=0}^{n} (S u^y d^{n-y} - K)^+ P(Y=y)$$

$$C = \frac{1}{(1+r)^n} \underset{\substack{\uparrow \\ RNP}}{\tilde{E}} \left[(S u^Y d^{n-Y} - K)^+ \right]$$

Price = discounted RN $\tilde{E}$

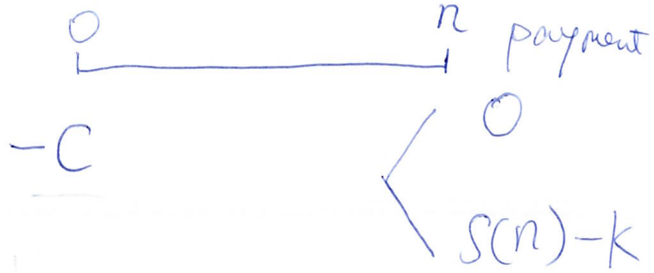
Proof. $S(n) \le K$ $\quad PV(\overset{\text{Pay off of}}{\text{call}}) = 0$

$S(n) > K$

$$PV(\text{payoff of Call}) = (S(n)-K)(1+r)^{-n}$$

$$PV(\text{payoff of Call}) = \frac{(S(n)-K)^+}{(1+r)^n}$$

$$PV(\text{return of Call}) = \frac{(S(n)-K)^+}{(1+r)^n} - C$$



Arbitrage Theorem:

$$\sum_{i_1,\cdots,i_n} r(i_1, i_2, \cdots, i_n) P(X_1 = i_1, \cdots, X_n = i_n) = 0 \quad \leftarrow \text{Eq (1)}$$

$$\Rightarrow \sum_{i_1,\cdots,i_n} \left[\frac{(S u^{\sum_{j=1}^n i_j} d^{n-\sum_{i=1}^n i_j} - K)^+}{(1+r)^n} - C \right] \cdot P(X_1 = i_1, \cdots, X_n = i_n) = 0$$

$$\Rightarrow \sum_{y=0}^{n} \left[\frac{(S u^y d^{n-y} - K)^+}{(1+r)^n} - C \right] P(Y = y) = 0$$

replace $X, \Sigma i$ with y

rearrangement

$$C = \frac{1}{(1+r)^n} \sum_{y=0}^{n} (Su^y d^{n-y} - k)^+ P(Y=y) = \frac{1}{(1+r)^n} \tilde{E}\left[(Su^Y d^{n-Y} - k)^+ \right]$$

$\boxed{//}$