RELATIVITY – MTH6132

PROBLEM SET 2

1. Let F and F' be two inertial frames in standard configuration with relative velocity v. Draw a combined spacetime diagram containing the (x, t) axes and (x', t') axes. Setting $c = 1$, and confirm that your diagram satisfies the following property: the x'-axis rotates up from x-axis at the same angle as the t'-axis rotates away from the t-axis. Can you prove this using the fact that c is constant to observers in any inertial reference frame?

2. Invert the Lorentz Transformations $t' = \gamma(t - vx/c^2)$, $x' = \gamma(x - vt)$ to solve for x and t in terms of x', t'. You should obtain the inverse Lorentz Transformations $t = \gamma(t' + vx'/c^2)$, $x = \gamma(x' + vt')$. In both formulae, $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

3. Show that the wave equation $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$ *does not* remain invariant under the 1–dimensional Galilean transformations. but that it *does* remain invariant under the Lorentz transformations

Hint: Recall that the chain rule for partial derivatives gives

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial x} = \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}.$$

Use these expressions twice to compute $\dfrac{\partial^2\phi}{\partial x^2}, \dfrac{\partial^2\phi}{\partial t^2}$.

4. (Extension of Problem 3) Starting from the Galilean transformation in the form $\underline{r}' = \underline{r} - \underline{v}t, \quad \underline{v} = (v_x, v_y, v_z)$, show that the scalar wave equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

does not remain invariant under these transformations.

Hint: recall that $t = t(t', x', y', z')$ and $x = x(t', x', y', z')$. Use the chain rule of partial differentiation to show that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v_x\frac{\partial}{\partial x'} - v_y\frac{\partial}{\partial y'} - v_z\frac{\partial}{\partial z'} = \frac{\partial}{\partial t'} - \underline{v}\cdot\nabla'$$

and similar formulae for $\frac{\partial}{\partial x}$ etc.

5. Starting from the Lorentz transformations between two frames F and F' in standard configuration (with F' moving with velocity of magnitude v relative to F) show

$$ct' - x' = \epsilon(ct - x), \qquad ct' + x' = \frac{1}{\epsilon}(ct + x)$$

where $\epsilon = \sqrt{\frac{1+v/c}{1-v/c}}$. Hence show that the combination of two LTs is a LT.

6. Consider F and F' in standard configuration. Let a standard clock be at rest in F' and consider two events in this clock at times t'_1 and t'_2 so that $\Delta t' = t'_2 - t'_1$. What is the time interval as measured by F? Draw and explain a spacetime diagram to accompany your calculations.

7. Consider F and F' in standard configuration. Let a rod of length $\Delta x'$ be at rest along the x' axis in F'. What is the length of this rod as measured by F? Draw and explain a spacetime diagram to accompany your calculations.

8. Charged pions are produced in an accelerator and emerge from the machine at a speed of $.996c$. Such unstable particles decay with a half-life of 1.8×10^{-8} seconds. According to an observer in the laboratory, how far can such particles travel before half of them decay? Perform a similar calculation *without* time dilation and compare your results.

9. A train travels along the x-axis past a platform at a speed v. A passenger at rest in the train holds a measuring ruler of length L parallel to the x-axis. Determine the speed at which the train must be travelling in order for the length of the ruler as measured by the observer at the platform to be $L/3$.

10. Consider two inertial systems of reference Joe and Moe. Moe moves with velocity v with respect to Joe. In Joe's frame of reference a ray of light is shot at $t = 0$ from $x = L$ towards the origin where a mirror reflects it back. The ray reaches $x = 2L$ at time t_1. Draw a spacetime of the situation as seen by Joe. Draw also the situation as seen by Moe who is moving with positive velocity $v < c$ along Joe's x-axis.

Further Exploration. Here are some more challenging and diverse problems to consider when reading the notes:

- Show that Maxwell's equations are not invariant under the GT but are invariant under the LT (**Hint**: consider the easier 1 and 2-dimensional cases first).
- Look up an article on the Michelson-Morley experiment and write down the main ideas behind construction of the interferometer.