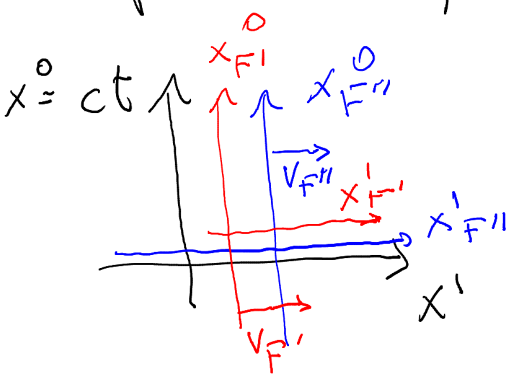
Tutorial Week 2

(1) Compose two Boost along x



where $v_{F'}$ is the velocity of F' with respect to F an $v_{F''}$ is the velocity of F'' with respect to F'. What is the boost connecting F'' to F?

According to Galilean relativity it is $v_{F''}+v_{F'}$.

$x'' = x' - v_{F''}t'$ $\quad t''=t'=t \Longrightarrow \quad x'' = x - v_{F'}t - v_{F''}t = x - (v_{F''}+v_{F'})t$

$x' = x - v_{F'}t$

The easiest approach is to use the matrix notation focusing on the plane (x^0, x^1)

$$L_{F''F} = \underbrace{\begin{pmatrix} \cosh\alpha'' & \sinh\alpha'' \\ \sinh\alpha'' & \cosh\alpha'' \end{pmatrix}}_{L_{F''F'}} \underbrace{\begin{pmatrix} \cosh\alpha' & \sinh\alpha' \\ \sinh\alpha' & \cosh\alpha' \end{pmatrix}}_{L_{F'F}}$$

where $\tanh\alpha'' = \frac{V_{F''}}{c}$ and $\tanh\alpha' = \frac{V_{F'}}{c}$. Then

$$L_{F''F} = \begin{pmatrix} \overbrace{\cosh\alpha''\cosh\alpha' + \sinh\alpha''\sinh\alpha'}^{\cosh(\alpha''+\alpha')} & \sinh(\alpha''+\alpha') \\ \underbrace{\sinh\alpha''\cosh\alpha' + \cosh\alpha''\sinh\alpha'}_{\sinh(\alpha''+\alpha')} & \cosh(\alpha''+\alpha') \end{pmatrix}$$

Thus the rapidities (rather than the velocities) sum

$$\alpha = \alpha'' + \alpha'$$

The boost relating F'' and F is

$$\frac{V}{c} = \tanh\alpha = \frac{\sinh(\alpha''+\alpha')}{\cosh(\alpha''+\alpha')} = \frac{\sinh\alpha''\cosh\alpha' + \sinh\alpha'\cosh\alpha''}{\cosh\alpha''\cosh\alpha' + \sinh\alpha''\sinh\alpha'} =$$

$$\frac{\tanh\alpha'' + \tanh\alpha'}{1 + \tanh\alpha'\tanh\alpha''} = \frac{1}{c}\,\frac{V_{F''} + V_{F'}}{1 + \frac{V_{F'}V_{F''}}{c^2}}$$

Check that in the $V_{F'}, V_{F''} \ll c$ limit, you reproduce the Galilean result.

(2) Consider a particle moving with velocity V' in the frame F', what is its velocity in the frame F (in the same case as (1))?

We can recycle (1) choosing F'' so as the particle is at rest at t', so

$$V = \frac{V' + v_{F'}}{1 + \frac{V' v_{F'}}{c^2}}$$

You can also use the definition of V

$$\frac{dx}{dt} = \frac{d\left(\gamma\left(x' + v_{F'} ct'\right)\right)}{d\left(\gamma\left(t' + \frac{v_{F'}}{c^2} x'\right)\right)} = \frac{\frac{dx'}{dt'} + v_{F'}}{1 + \frac{v_{F'}}{c^2}\frac{dx'}{dt'}}$$

Now $\frac{dx'}{dt'} = V'$ and $\frac{dx}{dt} = V$ so you get.

(3) Now suppose that the particle undergoes an acceleration a' at t', i.e. $\frac{dV'}{dt'} = a'$. What is the acceleration measured in F?

$$a = \frac{dV}{dt} = \frac{d}{d\left(\gamma\left(t' + \frac{v_{F'}}{c^2} x'\right)\right)} \left(\frac{V' + v_{F'}}{1 + \frac{V' v_{F'}}{c^2}}\right) =$$

$$\frac{1}{\gamma} \frac{1}{1 + \frac{v_{F'} V'}{c^2}} \frac{d}{dt'} \left(\frac{V' + v_{F'}}{1 + \frac{V' v_{F'}}{c^2}}\right)$$

Recall that $v_{F'}$ is constant, we have

$$a = \frac{1}{\gamma\left(1+\frac{V_F' V'}{c^2}\right)}\left[\frac{a'}{1+\frac{V_F' V'}{c^2}} - \frac{V'+V_F'}{\left(1+\frac{V_F' V'}{c^2}\right)^2}\frac{a' V_F'}{c^2}\right] =$$

$$= \frac{a'}{\gamma\left(1+\frac{V_F' V'}{c^2}\right)^3}\left[1+\cancel{\frac{V_F' V'}{c^2}} - \left(\cancel{V'}+V_F'\right)\frac{V_F'}{c^2}\right]$$

$$= \frac{a'}{\gamma^3\left(1+\frac{V_F' V'}{c^2}\right)^3}$$

(4) Start from the Lorentz boost

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$
$$x' = \gamma\left(x' - \frac{v}{c}ct\right)$$
$$y' = y', \quad z' = z$$

and guess the transformation for a general velocity $\underline{v}$ (not necessarily along x).

You should get

(a) $$ct' = \gamma\left(ct - \frac{\underline{v}\cdot\underline{x}}{c}\right)$$

(b) $$\underline{x}' = \underline{x} + \underline{v}\left[\frac{\underline{x}\cdot\underline{v}}{v^2}(\gamma-1) - \gamma t\right]$$

with $v^2 = \underline{v}\cdot\underline{v}$

$$\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

[...]

Consider the projector $P\underline{x} = \frac{\underline{x}\cdot\underline{v}}{v^2}\,\underline{v}$

$P\underline{x}$ is orthogonal to $\underline{v}$

$(P\underline{x})\cdot\underline{w} = \frac{\underline{x}\,\underline{v}}{v^2}(\underline{w}\cdot\underline{v}) = 0$ for any $\underline{w}$ such that $\underline{w}\,\underline{v}=0$

The orthogonal complement to $P\underline{x}$ is $\underline{x} - \frac{\underline{x}\cdot\underline{v}}{v^2}\underline{v}$ and this should not transform. Check: the orth. compl. to v of each side of (b) is

$$\underline{x}' - \frac{\underline{x}'\cdot\underline{v}}{v^2}\,\underline{v} = \underline{x} - \frac{\underline{x}\cdot\underline{v}}{v^2} + \overbrace{\left[\cdots\right]}^{\text{same as above}}\cancelto{\text{zero}}{\left(\underline{v} - \frac{\underline{v}\cdot\underline{v}}{v^2}\,\underline{v}\right)}$$

Instead the component along $\underline{v}$ $\left(x'_v \equiv \frac{\underline{x}\cdot\underline{v}}{v}\right)$ reads

$$x'_v = x_v + \left[\frac{\underline{x}\,\underline{v}}{v^2}\gamma - \frac{\underline{x}\,\underline{v}}{v^2} - \gamma t\right]\frac{\underline{v}\,\underline{v}}{v} =$$

$$= \cancel{x_v} + x_v\gamma - \cancel{x_v} - \gamma t v = \gamma\left(x_v - vt\right)$$

as expected!