Actuarial Mathematics II
MTH5125

Premiums

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Plan

- Based on Chapter 6, DHW
- Net premiums
- Present value of future loss random variable
- Expected value of net random future loss and net premiums
- Variance of net random future loss
- Gross (expense-loaded) premiums

Net random future loss

An insurance contract is an agreement between two parties:

- the insurer agrees to pay for insurance benefits
- in exchange for insurance premiums to be paid by the insured

Notations:

- $PVFB_0$ the present value, at time of issue, of future benefits to be paid by the insurer (benefit outgo).
- $PVFP_0$ the present value, at time of issue, of future premiums to be paid by the insured (premium income).

The insurer's **net random future loss** (the present value of future loss *random variable*):

$$L_0^n = PVFB_0 - PVFP_0$$

$$L_0^n = PV \text{ of benefits outgo} - PV \text{ of premium income}$$

Net random future loss

Example

An insurer issues a whole life insurance to a person aged 60, with sum insured S payable immediately on death (continuous). Premiums are payable annually in advance, ceasing at age 80 or on earlier death. The net annual premium is P. Write down the net future loss random variable, L_0^n for this contract.

Net random future loss

Example

The present value random variable for the benefit is $Sv^{T_{60}}$

The present value random variable for the premium income is , $P\ddot{a}_{\overline{\min(K_{60}+1,20)}|}$

Thus,

$$L_0^n = Sv^{T_{60}} - P\ddot{a}_{\overline{\min(K_{60}+1,20)}|}$$

The equivalence principle premium

The net premium, generically denoted by P, may be determined according to **the principle of equivalence:**

$$\begin{aligned} E\left(L_0^n\right) &= 0 \\ E\left(PVFB_0\right) &= E\left(PVFP_0\right) \\ EPV\ of\ benefit\ outgo &= EPV\ of\ premium\ income \end{aligned}$$

Actuarial value of future benefits = *Actuarial value of future income*

The equivalence principle premium: whole life insurance

The net annual premium

An insurer issues a whole life insurance (annual) to a life aged x, with sum insured S (payable at the end of the year of death). Premiums are payable *annually in advance* until death. Find the net annual premium is P if $S = 100,000$ and $x = 60$. Consider the Standard Life Table with $i = 5\%$.

$$\begin{aligned} L_0^n &= Sv^{K_x+1} - P\ddot{a}_{\overline{K_x+1}|}; E\left(L_0^n\right) = 0 \\ E\left(Sv^{K_x+1}\right) &= E\left(P\ddot{a}_{\overline{K_x+1}|}\right) \\ SA_x &= P\ddot{a}_x \end{aligned}$$

$$P = \frac{SA_x}{\ddot{a}_x}$$

$$P = 100,000\frac{0.29028}{14.9041} = 1947.652$$

The equivalence principle premium: whole life insurance

You can express the net annual premium:

- in terms of the annuity functions as:

$$P = S\frac{1-d\ddot{a}_x}{\ddot{a}_x} = S\left(\frac{1}{\ddot{a}_x} - d\right)$$

- in terms of the insurance annuity functions as:

$$P = S\frac{A_x}{\left(1-A_x\right)/d} = S\frac{dA_x}{1-A_x}$$

We used:

$$A_x = 1 - d\ddot{a}_x$$

$$d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i}$$

The equivalence principle premium: whole life insurance

The variance of the net random future loss

$$
\begin{aligned}
L_0^n &= Sv^{K_x+1} - P\ddot{a}_{\overline{K_x+1}|} \\
&= Sv^{K_x+1} - P\frac{1-v^{K_x+1}}{d} \\
&= \left(S + \frac{P}{d}\right)\left(v^{K_x+1}\right) - \frac{P}{d}
\end{aligned}
$$

$$
\begin{aligned}
V\left(L_0^n\right) &= \left(S + \frac{P}{d}\right)^2 V\left(v^{K_x+1}\right) \\
&= \left(S + \frac{P}{d}\right)^2 \left[{}^2A_x - (A_x)^2\right]
\end{aligned}
$$

Note - in Actuarial maths 1: ${}^2A_x$ was denoted with A_x^*

The equivalence principle premium: whole life insurance

The variance of the net random future loss
Once you found P you can work out the variance:

$$
\begin{aligned}
V\left(L_0^n\right) &= \left(S + \frac{P}{d}\right)^2 \left[{}^2A_x - (A_x)^2\right] \\
&= S^2 \left(1 + \frac{A_x}{d\ddot{a}_x}\right)^2 \left[{}^2A_x - (A_x)^2\right] \\
&= S^2 \left(1 + \frac{1 - d\ddot{a}_x}{d\ddot{a}_x}\right)^2 \left[{}^2A_x - (A_x)^2\right] \\
&= S^{\,2} \frac{{}^2A_x - (A_x)^2}{(d\ddot{a}_x)^2} = S \frac{{}^2A_x - (A_x)^2}{(1 - A_x)^2}
\end{aligned}
$$

The equivalence principle premium: endowment insurance

Consider an n-year endowment policy which pays S dollars at the end of the year of death or at maturity, issued to a life with exact age x. Net premium of P is paid at the beginning of each year throughout the policy term.

Some helpful relations

- Reminder:

$$\begin{aligned} A_x &= 1 - d\ddot{a}_x \text{ and } \bar{A}_x = 1 - \delta\ddot{a}_x \\ A_{x:\overline{n}|} &= 1 - d\ddot{a}_{x:\overline{n}|} \\ \bar{A}_{x:\overline{n}|} &= 1 - \delta\ddot{a}_{x:\overline{n}|} \end{aligned}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}$$

$$d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i}$$

The equivalence principle premium: endowment insurance

The net random future loss is:

$$L_0^n = Sv^{\min(K_x+1,n)} - P\ddot{a}_{\overline{\min(K_x+1,n)}|}$$

Expected net random loss:

$$\begin{aligned} E\left(L_0^n\right) &= SE\left(v^{\min(K_x+1,n)}\right) - PE\left(\ddot{a}_{\overline{\min(K_x+1,n)}|}\right) \\ &= SA_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|} \end{aligned}$$

Equivalence principle:

$$\begin{aligned} E\left(L_0^n\right) &= 0 \\ SA_{x:\overline{n}|} &= P\ddot{a}_{x:\overline{n}|} \end{aligned}$$

$$P = \frac{SA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = S\left(\frac{1}{\ddot{a}_{x:\overline{n}|}} - d\right)$$

The equivalence principle premium: endowment insurance

The variance of net random loss

$$
\begin{aligned}
L_0^n &= Sv^{\min(K_x+1,n)} - P\ddot{a}_{\overline{\min(K_x+1,n)}|} \\
&= Sv^{\min(K_x+1,n)} - P\frac{1 - v^{\min(K_x+1,n)}}{d} \\
&= \left(S + \frac{P}{d}\right) v^{\min(K_x+1,n)} - \frac{P}{d}
\end{aligned}
$$

$$
\begin{aligned}
V\left(L_0^n\right) &= \left(S + \frac{P}{d}\right)^2 V\left(v^{\min(K_x+1,n)}\right) \\
&= \left(S + \frac{P}{d}\right)^2 \left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right]
\end{aligned}
$$

The equivalence principle premium

Consider now that the sum insured is \$100,000, mortality table follows the Standard Ultimate Life Table with $i = 5\%, x = 50$ and $n = 20$.

The annual net premium is:

$$P = \frac{SA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = 100,000 \frac{A_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}}$$

Some helpful relations

Pure endowment: ${}_nE_x = v^n \ {}_np_x = v^n \ \frac{l_{x+n}}{l_x}$

Whole life insurance:

$$
\begin{aligned}
A_x &= A^1_{x:\overline{n}|} + \ v^n \ {}_np_x A_{x+n} \\
A_x &= A^1_{x:\overline{n}|} + \ {}_nE_x A_{x+n}
\end{aligned}
$$

Some helpful relations

$$\text{Term insurance: } A^{1}_{x:\overline{n}|} = A_x - \ {}_nE_x A_{x+n}$$

$$\text{Endowment insurance} A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + \ {}_nE_x$$

$$\text{Endowment insurance: } A_{x:\overline{n}|} = A_x -_n E_x A_{x+n} +_n E_x$$

Note that for the continous endowment insurance when the benefit paid is immediately on death:

$$\bar{A}_{x:\overline{n}|} = \frac{i}{\delta}\left(A_x -_n E_x A_{x+n}\right) +_n E_x$$

Note the term $\frac{i}{\delta}$ to account for difference in the time of payment (using UDD - uniform distribution of deaths)

Some helpful relations

$$
\begin{aligned}
A_{50:\overline{20}|} &= A_{50} - {}_{20}E_{50} A_{70} + {}_{20}E_{50} \\
& 0.18913 - 0.34824 \times 0.42818 + 0.3482410
\end{aligned}
$$

Some helpful relations

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - \ v^n \ {}_np_x \ddot{a}_{x+n}$$

$$a_{x:\overline{n}|} = a_x - \ v^n \ {}_np_x a_{x+n}$$

$$a_x = \ddot{a}_x - 1$$

Some helpful relations

$$\ddot{a}_{50:\overline{20}|} = \frac{1 - A_{50:\overline{20}|}}{0.05/1.05} = 11.16779$$

$$P = \frac{SA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = 100,000 \frac{A_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}} = 4192.42$$

Gross (expense-loaded) premiums

Investment-related expenses (e.g. analysis, cost of buying, selling, servicing)

Insurance-related expenses:

- acquisition (agents' commission, underwriting, preparing new records)
- maintenance (premium collection, policyholder correspondence)
- general (research, actuarial, accounting, taxes)
- settlement (claim investigation, legal defense, disbursement)

Gross (expense-loaded) premiums

Most life insurance contracts incur large losses in the first year because of large **first year expenses**:

- agents' commission
- preparing new policies, contracts
- records administration
- these large losses are hopefully recovered in later years.

Renewal expenses are expenses used for maintaining and continuing a policy

- percentage of premium
- per policy amount
- combination of the two above

Termination expenses: when a policy expire (death or maturity) - very small

Gross (expense-loaded) premiums

New business strain

- First year premium is insufficient to cover first year expenses.

Consequence:

- The insurer needs funds (which he borrows from shareholders) in order to be able to sell new policies
- This loan is gradually paid off by the policyholder via expense loadings in his future premiums

Triggers:

- a high initial agent commission,
- an aggressive portfolio growth strategy.

Gross premium calculations

Treat expenses as if they are a part of benefits.
The gross random future loss at issue is

$$L_0^g = PVFB_0 + PVFE_0 - PVFP_0$$

$PVFE_0$: the present value random variable associated with future expenses incurred by the insurer.
Equivalence principle:

$$E\left(L_0^g\right) = E\left(PVFB_0\right) + E\left(PVFE_0\right) - E\left(PVFP_0\right) = 0$$

$$E\left(PVFP_0\right) = E\left(PVFB_0\right) + E\left(PVFE_0\right)$$

EPV of gross premiums $=$ EPV of benefits $+$ EPV of expenses

Example

An insurer issues a 20-year annual premium endowment insurance with sum insured $\$100,000$ to an individual aged 30. The insurer incurs initial expenses of $\$2,000$ plus 50% of the first premium, and renewal expenses of 2.5% of each subsequent premium. The death benefit is payable immediately on death (continuous).

(a)Write down the gross future loss random variable.

(b) Calculate the gross premium using the Standard Select Survival Model with 5% per year interest.

Gross premium calculations

$$S = 100,000,\ x = 30,\ P \text{ the annual gross premium}$$

Gross random future loss L_0^g:

$L_0^g = Sv^{\min(T_x,n)} + 2,000 + 0.5P + 0.025Pa_{\overline{\min(K_x+1),n)}|} - P\ddot{a}_{\overline{\min(K_x+1),n)}|}$

$L_0^g = Sv^{\min(T_x,n)} + 2,000 + 0.5P + 0.025P\left(\ddot{a}_{\overline{\min(K_x+1),n)}|} - 1\right) - P\ddot{a}_{\overline{\min(K_x+1),n)}|}$

Gross premium calculations

$$L_0^g = Sv^{\min(T_x,n)} + 2,000 + 0.025P\ddot{a}_{\overline{\min(K_x+1),n)}|} + 0.475P - P\ddot{a}_{\overline{\min(K_x+1),n)}|}$$

Gross premium calculations

$$S = 100,000,\ x = 30,\ P \text{ the annual gross premium}$$

Equivalence principle:

$$E\left(L_0^g\right) = 0$$

$$S\bar{A}_{30:\overline{20}|} + 2,000 + 0.025P\ddot{a}_{30:\overline{20}|} + 0.475P - P\ddot{a}_{30:\overline{20}|} = 0$$

Gross premium calculations

Expected present value of premium income:

$$E\left(P\ddot{a}_{\overline{\min(K_x+1),n)}|}\right) = P\ddot{a}_{30:\overline{20}|} = 13.04098P$$

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}$$

Gross premium calculations

$$A_{x:\overline{n}|} = A_x - {}_nE_x A_{x+n} + {}_nE_x$$
$$A_{30:\overline{n}|} = A_{30} - {}_{20}E_{30} A_{50} + {}_{20}E_{30}$$

$$A_{30:\overline{n}|} = 0.07698 - 0.372548 \times 0.18931 + 0.372548 = 0.379001$$

$$\ddot{a}_{30:\overline{20}|} = \frac{1 - A_{30:\overline{20}|}}{0.05/1.05} = 13.04098$$

Gross premium calculations

Expected present value of the expenses

$$2,000 + 0.025P\ddot{a}_{30:\overline{20}|} + 0.475P = 2,000 + 0.801025P$$

Expected present value of the death benefit (using UDD - uniform distribution of deaths):

$$\begin{aligned} S\bar{A}_{30:\overline{n}|} &= 100,000\left(\frac{i}{\delta}A^{1}_{30:\overline{n}|} +_{20} E^{30}\right) \\ &= 100,000\left(\frac{i}{\delta}\left(A_{30:\overline{n}|} -_{20} E^{30}\right) +_{20} E^{30}\right) \\ &= 37,912.16 \end{aligned}$$

Gross premium calculations

Example

An insurer issues a 20-year annual premium endowment insurance with sum insured $\$100,000$ to an individual aged 30. The insurer incurs initial expenses of $\$2,000$ plus 50% of the first premium, and renewal expenses of 2.5% of each subsequent premium. The death benefit is payable immediately on death (continuous).

$$P = \frac{37,912.16 + 2,000}{13.04098 - 0.801025} = 3260.809$$

Premium calculations: Profit

The equivalence principle does not allow explicitly for a loading for profit.

Since issuing new policies generally involves a loan from the shareholders to cover from new business strain - need for sufficient income for shareholders to make an adequate rate of return.

Premium calculations: Profit

Each individual policy sold will generate a profit or a loss.

- for each individual policy the experienced mortality rate in any year can take only the values 0 or 1.
- the expected outcome (ex-ante) under the equivalence principle is zero profit (assuming no margins), the actual outcome for each individual policy will either be a profit or a loss (ex-post).

For the actual profit from a group of policies to be close to the expected profit, insurer sells a large number of individual contracts, whose future lifetimes can be regarded as statistically independent:

- the losses and profits from individual policies are combined.

Premium calculations: Profit

Consider a life who purchases a one-year term insurance with sum insured $1000 payable at the end of the year of death. Let us suppose that the life is subject to a mortality of rate of 0.01 over the year, that the insurer can earn interest at 5% per year, and that there are no expenses.
Base on the equivalent principle the premium is:

$$P = 1,000 \times \frac{0.01}{1.05} = 9.52$$

Why?

Premium calculations: Profit

The quivalence principle is:

$$\begin{aligned} E\left(L_0^n\right) &= E\left(Sv^{\min(K_x+1,1)} - P\right) = 0 \\ P &= E\left(Sv^{\min(K_x+1,1)}\right) \end{aligned}$$

$$P = SA^1_{x:\overline{1}|} = Sv^1 \ {}_1|q_x$$

$$P = 1000 \times \frac{1}{1+0.05} \times 0.01 = 9.52$$

Premium calculations: Profit

The net random loss:

$$
\begin{aligned}
L_0^n &= \begin{cases} 1,000v - P = 942.86 & \text{if } T_x \leq 1 \text{ with prob. } 0.01 \\ -P = -9.52 & \text{if } T_x > 1 \text{ with prob. } 0.99 \end{cases} \\
E\left(L_0^n\right) &= 0
\end{aligned}
$$

Probability of profit: 0.99 vs. Probability of loss 0.01

Premium calculations: Profit

Insurer issues a large number of policies, so that the overall proportion of policies becoming claims will be close to the assumed proportion of 0.01.

Suppose the insurer were to issue 100 such policies to independent lives.

- if all lives survive for the year, then the insurer makes a profit.
- if one life dies, there is no profit or loss.
- if more than one life dies, there will be a loss on the portfolio.

Let D denote the number of deaths in the portfolio, so that $D \sim B(100, 0.01)$.

Probability that the profit on the whole portfolio is greater than or equal to zero is

$$Pr[D \leq 1] = 0.73576$$

Compare to 99% for the individual contract.

If number of policies increases to ∞, $Pr[D \leq 1]$ approaches 0.5.

Premium calculations: Profit

While the probability of loss is increasing with the portfolio size, the probability of very large aggregate losses (relative to total premiums) is much smaller for a large portfolio, since there is a balancing effect from diversification of the risk amongst the large group of policies.

The portfolio percentile premium principle

Suppose insurer issues a portfolio of N identical and independent policies where the PV of loss-at-issue for the i-th policy is $L_{0,i}$. The total loss at issue (total portfolio aggregate future loss) is:

$$L = \sum_{i=1}^{N} L_{0,i}$$

$$\begin{aligned} E(L) &= \sum_{i=1}^{N} E(L_{0,i}) = NE(L_{0,i}) \\ E(L) &= \sum_{i=1}^{N} V(L_{0,i}) = NV(L_{0,i}) \end{aligned}$$

The portfolio percentile premium principle

The portfolio percentile premium principle sets a premium so that there is a specified probability α, that the total future loss is negative:

$$P[L < 0] = \alpha$$

Now, if N is sufficiently large (say, greater than around 30), the central limit theorem tells us that L is approximately normally distributed, with mean $E[L] = NE[L_{0,1}]$ and variance $V[L] = NV[L_{0,1}]$.

The portfolio percentile premium principle

$$
\begin{aligned}
P[L &< 0] = P\left(\frac{L - E(L)}{\sqrt{V(L)}} < \frac{-E(L)}{\sqrt{V(L)}}\right) \\
&= \Phi\left(\frac{-E(L)}{\sqrt{V(L)}}\right) = \alpha
\end{aligned}
$$

which implies that:

$$
\frac{E(L)}{\sqrt{V(L)}} = -\Phi^{-1}(\alpha)
$$

Φ is the cumulative distribution function of the standard normal distribution.