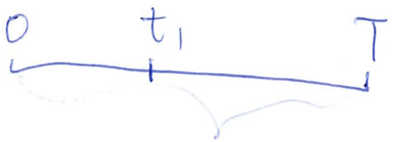
Advice:

1: Draw timelines



2. Every equation tells a story

Big picture:

Two seperate ~~branches~~ branches of Finance:

Q World – Risk Neutral Prob → MTH 6112
e.g. derivative pricing

P world – Real world Prob → MTH 6113
e.g. risk & Portfolio management

1. Wiener Process, Brownian Motion, Geometric Brownian Motion

W1 D

1.1. Definitions

Def 1.1

A stochastic process, $Y(t)$, is a collection of r.v.s $\{Y(t)\}_{t\geq 0}$

i.e. a r.v. $Y(t)$, for each $t\geq 0$.

Model: time-dependent random quantities

e.g. the most basic stochastic process is Brownian motion

Def 1.2: Wiener process / Standard Brownian motion

~ is a stochastic process $W(t)$ s.t.

① $W(0)=0$

② For any $t<T$ the r.v $W(T)-W(t)$ is normal with mean 0 and variance $T-t$.

$W(T)-W(t) \sim N(0, T-t)$

③ 

For any $0 < t_1 < t_2 \cdots < t_n$

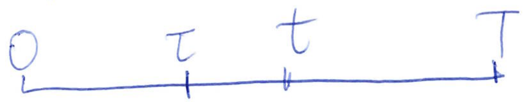
the increments $W(t_1)$, $W(t_2) - W(t_1)$, $\cdots$, $W(t_n) - W(t_{n-1})$ are independent r.v.s.

④ The function $t \mapsto W_t$ is continuous.

Remarks:

① $W(t_1)$ $\quad W(t_1) - W(0)$ $\quad$ Prop ①

② Prop ③:
For any $t < T$, the r.v $W(T) - W(t)$ is independent of the trajectory $W(\tau)$ $\quad 0 \le \tau \le t$



$\rightarrow$ the Markov property

Story: the future $\perp$ past

③ Prop ④ assume

Def 1.3

Brownian motion (BM)

- drift parameter μ
- volatility parameter σ

$$Y(t) = \mu t + \sigma W(t)$$

Proposition 1.1

① $Y(0) = 0$

② For ant $t < T$ the r.v $Y(T) - Y(t)$ is normal

$$Y(T) - Y(t) \sim N(\mu(T-t), \sigma^2(T-t))$$

$$Y(T) \sim N(\mu T, \sigma^2 T)$$

③ The increments $Y(t_1), Y(t_2) - Y(t_1), \cdots, Y(t_n) - Y(t_{n-1})$ are independent R.V.s

Proof: Hints:

Write $Y \leftarrow W$

① $W(0)=0$

② Use properties of W, normal distribution.

③ Use the def of independent variables

Remarks:

$Y(t) \sim N(\mu t, \sigma^2 t)$ can be negative.

$Y(T)-Y(t) \perp Y(t)$

Def. 1.4 Geometric Brownian Motion (GBM)

A ~~GBM~~ GBM with drift μ, volatility σ, starting value S,

is a process $S(t)$ given by

$$S(t) = S e^{\mu t + \sigma W(t)},$$

$W(t)$ is the Wiener process.

Remarks

① $W(0)=0$, $S(0)=S$

② $S(t)=Se^{Y(t)}$, $Y(t)$ BM

$\Rightarrow \ln S(t) = \ln S + Y(t) = \ln S + \mu t + \sigma W(t)$

③ Different defs of $W(t)$

1.2 Covariance of the Wiener process

$W(t) = W_t$

Lemma 1.1

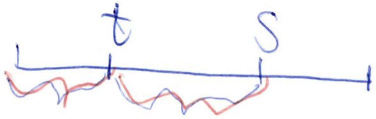
If W_t is the Wiener process then $Cov(W_t, W_s) = \min(t,s)$

Proof: $cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

$E(W_t) = E(W_s) = 0$

$Cov(W_t, W_s) = E(W_t W_s)$

If $t \le s$:



$$\mathrm{Cov}(W_t, W_s) = E(W_t W_s) = E\left[W_t(\underbrace{W_s - W_t} + W_t)\right]$$

$$= \underbrace{E\left[W_t(W_s - W_t)\right]}_{=0} + E\left[W_t^2\right]$$

Use Property ③ of Wiener Process.

W_t and $W_s - W_t$ are independent,

we have $E[W_t(W_s - W_t)] = E(W_t) \cdot E(W_s - W_t) = 0$

$$\mathrm{cov}(W_t, W_s) = \underline{E(W_t^2) = \mathrm{Var}(W_t)}$$

Hint: $E(W_t) = 0$

$$= t$$

$$\boxed{\begin{aligned} &\mathrm{Var}(W_t) \\ &= E(W_t^2) - (\underbrace{E(W_t)}_{=0})^2 \\ &= E(W_t^2) \end{aligned}}$$

If $t > s$ $\quad \mathrm{Cov}(W_t, W_s) = s$

$\mathrm{Cov}(W_t, W_s) = \min(t, s)$. ▨

1.3 Calculations with Normal r.v.s.

Proposition 1.2

If X is standard Normal r.v. with mean 0 and variance 1,

$X \sim N(0,1)$

For any $\mu \in \mathbb{R}$ and $\sigma > 0$, we have

$Y = \mu + \sigma X \sim N(\mu, \sigma^2)$

Proof:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Goal: prove $f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$

$f_Y(y) = F_Y'(y)$, where $F_Y(y) = P(Y \le y)$

$$F_Y(y) = P(\sigma X + \mu \le y) = P\left(X \le \frac{y-\mu}{\sigma}\right) = F_X\left(\frac{y-\mu}{\sigma}\right)$$

$$f_Y(y) = \frac{d}{dy} F_X\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right)$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}} \quad \square$$

Transformation Formula: $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$.

$$g^{-1}(y) = \frac{y-\mu}{\sigma}$$

Corollary 1.1

If $Y \sim N(\mu, \sigma^2)$ and $Z = aY+b$, the $Z \sim N(a\mu+b, a^2\sigma^2)$

Proof: Use Proposition 1.2, $Y = \mu + \sigma X$, $X \sim N(0,1)$

$$Z = aY + b = a(\sigma X + \mu) + b = a\sigma X + (a\mu + b)$$

By Prosition 1.2. $Z \sim N(a\mu+b, a^2\sigma^2)$

$$W_t \sim N(0, t)$$

1.4 The expectation of the GBM

Goal of this section: $E(S(t))$

Remark:

$Y(t) = \mu t + \sigma W(t)$, $W(t) \sim N(0, t)$ $\quad W_{T} \sim N(0, T)$

$\Rightarrow Y(t) \sim N(\mu t, \sigma^2 t)$. $\leftarrow$ Corollary 1.1

$\cancel{E(Y)(}\ E(Y(t)) = \mu t$, $Var(Y(t)) = \sigma^2 t$

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx \quad \text{if } \int_{-\infty}^{\infty} |g(x)| f_x(x)\, dx < \infty$$

e^{tx}

Theorem 1.1

If $S(t)$ is GBM with drift μ and volatily σ, then

$$E(S(t)) = S(0) e^{\mu t + \frac{\sigma^2}{2} t}$$

Proof:

$$S(t) = S(0)\exp\left(\mu t + \sigma W(t)\right) \leftarrow \text{def GBM}$$

$$E\left(S(t)\right) = S(0)\,E\left[e^{\mu t + \sigma W(t)}\right] = S(0)e^{\mu t}\,\underbrace{E\left[e^{\sigma W(t)}\right]}$$

Goal: $E\left[e^{\sigma W(t)}\right]$

$W(t) \sim N(0,t)$ $\qquad f_{W_t}(x) = \dfrac{1}{\sqrt{2\pi t}}\exp\left(-\dfrac{x^2}{2t}\right)$

$f_X(x)$

$$E\left(e^{\sigma W(t)}\right) = \int_{-\infty}^{+\infty} e^{\sigma x}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)dx$$

X

$W(t) = X$

$e^{\sigma X}$

$$E\left(e^{\sigma X}\right) = \int_{-\infty}^{+\infty} e^{\sigma X}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)dX$$

$$= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{\sigma x - \frac{x^2}{2t}}\,dx$$

Completion of squares:

$$-\frac{x^2}{2t} + \sigma X = -\frac{x^2 - 2t\sigma x}{2t} = -\frac{(x - t\sigma)^2 - t^2\sigma^2}{2t} = -\frac{(x - t\sigma^2)}{2t} + \frac{t\sigma^2}{2}$$

$$E\left(e^{\sigma W(t)}\right) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-t\sigma)^2}{2t} + \frac{t\sigma^2}{2}\right) dx$$

$$= \frac{e^{\frac{t\sigma^2}{2}}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-t\sigma)^2}{2t}} \, dx$$

$$= e^{\frac{t\sigma^2}{2}} \int_{-\infty}^{+\infty} \underbrace{\boxed{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-t\sigma)^2}{2t}}}} \, dx$$

PDF of $N(\cancel{0}\, t\sigma, t)$

$$\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$$

$$= e^{\frac{t\sigma^2}{2}}$$

$$E(S(t)) = S(0)\, e^{\mu t + \frac{\sigma^2 t}{2}} \quad \blacksquare$$

1.5 The moments of the Wiener process

$E(W_t^2) \quad E(W_t^4)$

$$\begin{cases} E\left(e^{\sigma W(t)}\right) = e^{\frac{t\sigma^2}{2}} & (2) \\ e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} & (3) \end{cases}$$

Lemma 1.2

For any integer $j \geqslant 1$

$$E\left(W_t^{2j}\right) = \frac{(2j)!}{j!\,2^j} t^j \quad \text{and} \quad E\left(W_t^{2j-1}\right) = 0.$$

Proof: Eq (3)

$$e^{\sigma W_t} = \sum_{n=0}^{\infty} \frac{\sigma^n W_t^n}{n!} \qquad z = \sigma W_t$$

$$E\left(e^{\sigma W_t}\right) = E\left[\sum_{n=0}^{\infty} \frac{\sigma^n W_t^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{\sigma^n E\left(W_t^n\right)}{n!}$$

Eq (3) again ||

$$e^{\frac{\sigma^2 t}{2}} = \sum_{j=0}^{\infty} \frac{\sigma^{2j} t^j}{2^j j!}$$

$$z = \frac{\sigma^2 t}{2}$$

$$j = n$$

Use Eq (2)

$$\sum_{n=0}^{\infty} \frac{\sigma^n E\left(W_t^n\right)}{n!} = \sum_{j=0}^{\infty} \frac{\sigma^{2j} t^j}{2^j j!}$$

If $n = 2j-1$ then $E(W_t^{2j-1}) = 0$

If $n = 2j$ then $\dfrac{E(W_t^{2j})}{2j!} = \dfrac{t^j}{2^j j!}$

Lemma 1.3

The 4th moment of W_t and the variance of W_t^2 are given by

$$E(W_t^4) = 3t^2, \quad Var(W_t^2) = 2t^2$$

Proof: Use L1.2

$j=1 \quad W_t^2 \qquad j=2 \quad W_t^4$

$$E(W_t^2) = \frac{(2)!}{1!\,2} t = t$$

$j=1$

$$E(W_t^4) = \frac{4!}{2!\,2^2} t^2 = 3t^2$$

$j=2$