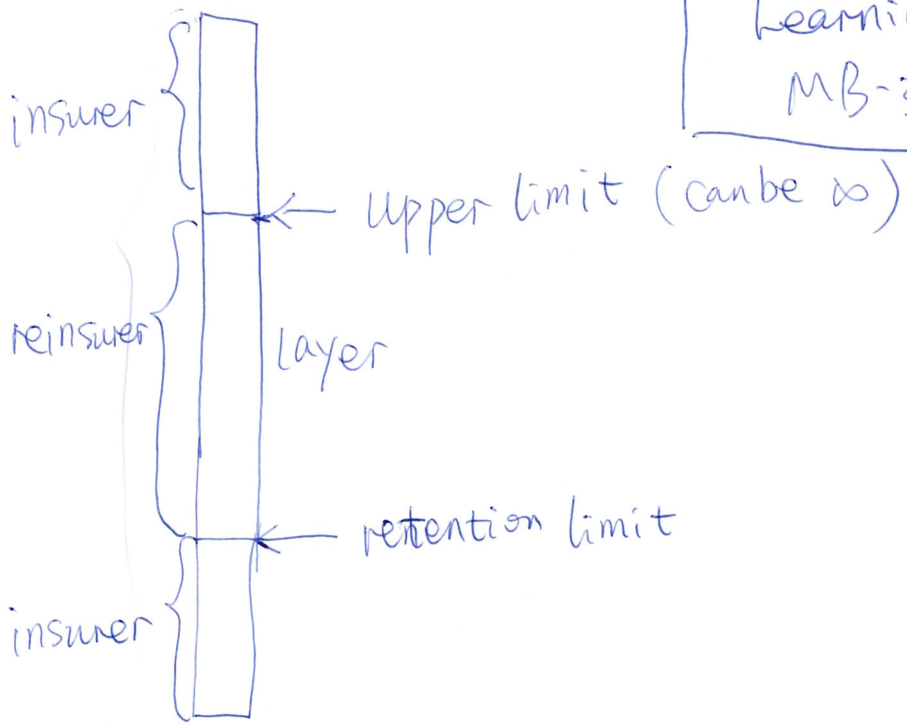
Week 2 Reinsurance

Thur 12:00-13:00
Learning Cafe W2-12 x7
MB-314 W1



Slide 6

① XOL

$X > M$

↓

a particular claim

M: excess point
retention

② Stop loss

$X' > M$

↓

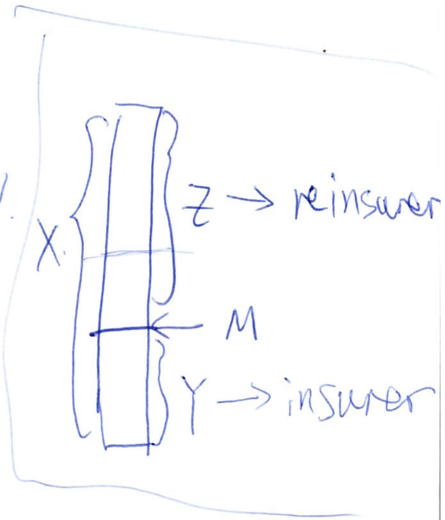
a specific group

Notation:

X: gross claim amount r.v.

Y: net claim amount

Z: the amount paid by the reinsurer



$X = Y + Z$

XOL Reinsurance

I: $Y = \begin{cases} X & \text{if } X \le M \\ M & \text{if } X > M \end{cases}$

R: $\boxed{Z = X - Y}$

$$Z = \begin{cases} 0 & \text{if } X \le M \\ X - M & \text{if } X > M \end{cases}$$

$$Y + Z = \begin{cases} 0 + X = X & \text{if } X \le M \\ M + X - M = X & \text{if } X > M \end{cases}$$

With Reinsurance,

Mean $X \to Y$ reduced

Var reduced

Upper limit on large claims

Insurer's perspective

Mean without Reinsurance

$$E(X) = \int_0^\infty x f(x)\,dx = \int_0^M x f(x)\,dx + \int_M^\infty x f(x)\,dx$$

Mean with Reinsurance

$$E(Y) = \int_0^M x f(x)\,dx + M P(X > M)$$

$X > M$

$$\int_M^\infty M f(x)\,dx$$

$$E(X) \ge E(Y)$$

MGF of Y:

$$M_Y(t) = E(e^{tY}) = \int_0^M e^{tx} f(x)\,dx + \int_M^\infty e^{tM} f(x)\,dx$$

$$= \int_0^M e^{tx} f(x)\,dx + e^{tM} P(X > M)$$

Reinsurer's Perspective

$$Z = \begin{cases} 0 & \text{if } X \le M \\ X-M & \text{if } X > M \end{cases}$$

$$E(Z) = \int_M^\infty (x-M) f(x)\,dx$$

$$M_Z(t) = E(e^{tZ}) = \int_0^M e^{t0} f(x)\,dx + \int_M^\infty e^{t(x-M)} f(x)\,dx$$

Reinsurer: record $>M$ $\longrightarrow$ truncated distribution

$W = X - M,\ X > M$ $\qquad W \neq Z$

PDF of W: $g(w)$

$$P(W<w) = P(X<w+M \mid X>M)$$

$$= \frac{P(X<w+M \text{ and } X>M)}{P(X>M)}$$

$$= \frac{P(M<X<w+M)}{P(X>M)}$$

since $P(a<X<b) = \int_a^b f(x)\,dx = F(b)-F(a)$

$$= \frac{F(w+M)-F(M)}{1-F(M)}$$

Differentiating wrt w

PDF $g(w) = \dfrac{f(w+M)}{1-F(M)}$, $w>0$

E.g. Claim $\sim N(500, 400)$

$M = 550$

Find: mean of reinsurer $E(Z)$ $\quad Z = \begin{cases} 0 & \text{if } X < M \\ X - M & \text{if } X \geq M \end{cases}$

A:
$$E(Z) = \int_0^{550} 0 \cdot f_X(x)\,dx + \int_{550}^{\infty} (x-550)\, f_X(x)\,dx$$

$f_X(x)$ is the PDF of $N(500, 400)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma^2} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

$u = \dfrac{x-500}{20}$ in the second integral,

$$E(Z) = \int_{2.5}^{\infty} (20u - 50) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\,du$$

$= 0.04$

MGF of Gamma distribution

Slide 17, Week 1.

Learning Café

Thur 12:00 – 13:00

Prove: $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$

Proof: $M_X(t) \overset{\text{def}}{=} \underline{E[e^{tX}]} = \int_0^\infty e^{tx} \cdot f_X(t)\, dx = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x} dx$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} \boxed{\lambda^\alpha} x^{\alpha-1} e^{-(\lambda-t)x} dx$$

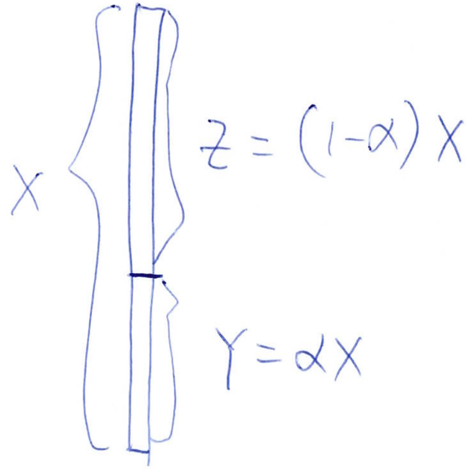
$$= \left(\frac{\boxed{\lambda}}{\lambda-t}\right)^\alpha \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha)} (\lambda-t)^\alpha x^{\alpha-1} e^{-(\lambda-t)x} dx}_{=1}$$

PDF of a Gamma $(\alpha, \lambda-t)$ distribution

$$= \left(\frac{\lambda}{\lambda-t}\right)^\alpha = \left(1-\frac{t}{\lambda}\right)^{-\alpha}, \quad t<\lambda$$

Proportional reinsurance

$$\begin{cases} Y = \alpha X \\ Z = (1-\alpha) X \end{cases}$$



retention level $\begin{cases} \text{XOL} \quad M \text{ fixed amount} \\ \text{proportional} \quad \alpha \quad \text{proportion} \end{cases}$

Q: $\boxed{\text{Reinsurer's}}$ payments. Proportional reins Slide 15 W2

values: £ 000s

4.6, 6.8, 22.9, 1.4, 3.8, 10.2, 19.4, 32.1

$X \sim \text{Gamma}(\alpha, \lambda)$, retained proportion is 80%

Find $\boxed{\overset{\text{reinsurer}}{Z \sim ?}}$, α, λ using method of moments reinsurer claim amount

$Z = 0.2X \Rightarrow X = 5Z$

Goal: PDF of Z

$$\int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^\alpha \underset{5z}{x}^{\alpha-1} e^{-\lambda \overset{5z}{x}} d\underset{5z}{x} \qquad X \sim \text{Gamma}(\alpha, \lambda)$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^\alpha (5z)^{\alpha-1} e^{-5\lambda z}\, 5\, dz \qquad d5z = 5dz$$

$$= \int_0^\infty \underbrace{\frac{1}{\Gamma(\alpha)} (5\lambda)^\alpha z^{\alpha-1} e^{-5\lambda z}}_{\hookrightarrow \text{PDF Gamma}(\alpha, 5\lambda)} dz \qquad \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x}$$

$Z \sim \text{Gamma}(\alpha, 5\lambda)$, population mean $= \frac{\alpha}{5\lambda}$, population Var $= \frac{\alpha}{25\lambda^2}$

Goal: estimate α, λ

$\bar{x} = \frac{\sum x_i}{8} = 12.65 \leftarrow$ sample mean

$s^2 = \frac{\sum x_i^2}{8} - \bar{x}^2 = 104.86 \leftarrow$ sample var

$\begin{cases} \text{Sample mean} = \text{population mean} \\ \text{Sample var} = \text{population var} \end{cases}$

$$\begin{cases} 12.65 = \dfrac{\alpha}{5\lambda} & ① \\ 104.86 = \dfrac{\alpha}{25\lambda^2} & ② \end{cases}$$

$\alpha = 1.526, \quad \lambda = 0.024$

Useful integral formulae

① ~~Logn~~ Lognormal

If $f_X(x)$: PDF of LogNormal (μ, σ^2)

then: $$\int_L^U x^k f_X(x)\,dx = e^{k\mu + \frac{1}{2}k^2\sigma^2}\left[\Phi(U_k) - \Phi(L_k)\right] \quad *$$

where: $L_k = \dfrac{\ln L - \mu}{\sigma} - k\sigma, \quad U_k = \dfrac{\ln U - \mu}{\sigma} - k\sigma$

$\Phi(z)$: ~~distribution function~~ CDF of the standard normal distribution

W2 ⑧

Proof:

$$*:\ \text{LHS} = \int_L^U x^k f_X(x)\,dx = \int_L^U x^k \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{L_k}^{U_k} e^{k(\mu+\sigma t+k\sigma^2)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t+k\sigma)^2} dt$$

$$t = \frac{\ln x-\mu}{\sigma} - k\sigma$$

$$= \int_{L_k}^{U_k} \frac{1}{\sqrt{2\pi}} e^{k\mu+k\sigma t+k^2\sigma^2} e^{-\frac{1}{2}t^2-k\sigma t-\frac{1}{2}k^2\sigma^2} dt$$

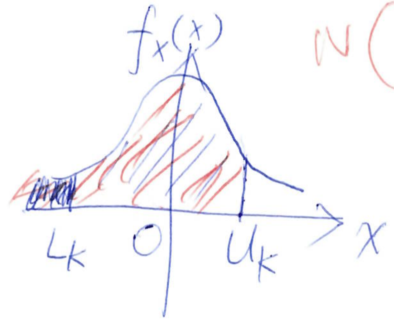
$$= e^{k\mu+\frac{1}{2}k^2\sigma^2} \int_{L_k}^{U_k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= e^{k\mu+\frac{1}{2}k^2\sigma^2}\left[\Phi(U_k) - \Phi(L_k)\right]$$

PDF $\sim N(\ ,\ )$ Standard Normal $N(0,1)$

$\Phi(\cdot)$

CDF of $N(0,1)$



Useful integral formulae

Normal distribution

If $f_X(x)$ is the PDF of the $N(\mu, \sigma^2)$

then $$\int_L^U x f_X(x)\,dx = \mu\left[\Phi(U') - \Phi(L')\right] - \sigma\left[\phi(U') - \phi(L')\right]$$

where $L' = \frac{L-\mu}{\sigma}$

$U' = \frac{U-\mu}{\sigma}$

$t = \frac{X-\mu}{\sigma}$

$\phi(\cdot)$ PDF of $N(0,1)$

Inflation

Assume: claim distribution remain constant over time

Q: X are inflated by a factor of k,
M remains fixed. for XOL

A: $X \rightarrow \underline{\underline{kX}}$ ~~(k>1)~~

$\uparrow$ claim amount

Y: amount paid by the insurer

$$Y = \begin{cases} kX & \text{if } kX \le M \\ M & \text{if } kX > M \end{cases}$$

Mean $E(Y) = \underbrace{\int_0^{\frac{M}{k}} kx f(x)\,dx}_{\substack{kx \le M \\ x \le \frac{M}{k}}} + \underbrace{MP\left(x > \frac{M}{k}\right)}_{\substack{kX > M \\ X > \frac{M}{k}}}$

$E(Y)$: with inflation
$E(Y')$: without inflation

$E(Y) \neq kE(Y')$

$M \rightarrow kM$

$E(Y) = kE(Y')$

M

Estimation when the sample in censored

$x_1, x_2, \underline{M}, x_3, \underline{M}, x_4, x_5$

Reinsurer – truncated

M

Insurer – censored

Estimation of censored sample: MLE

~~Stip~~ split the L into two parts:

① $x_1, x_2, \cdots, x_n$ are recorded exactly

$$L_1(\underline{\theta}) = \prod_{1}^{n} f(x_i; \underline{\theta})$$

② → reinsurer $M, M, \cdots$

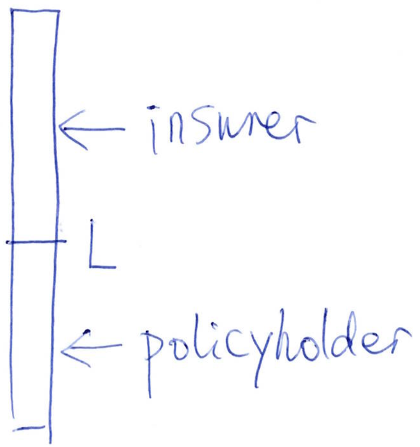
$$L_2(\underline{\theta}) = \prod_{1}^{m} P(X>M) = [1-F(M; \underline{\theta})]^m$$

CDF

$$L(\underline{\theta}) = \prod_{1}^{n} f(x_i; \underline{\theta}) \times [1-F(M, \underline{\theta})]^m$$

Assumption: claims are independent

Policy excess



X : Loss amount

L : limit / excess

Y : insurer

$$Y = \begin{cases} 0 & \text{if } X \le L \\ X - L & \text{if } X > L \end{cases}$$

Risk Models – basic model   Slide 34

$\boxed{X}$ – Single claim / loss

$\boxed{S}$ – aggregate claims paid by the insurer in the year in respect to a certain risk

Collective risk models

$$S = \sum_{i=1}^{N} X_i$$

X_i : amount of the i-th claim

N : claim numbers

Model claim numbers and claim amounts separately.

$S \sim$ Compound distribution

N (discrete) $\sim$

X_i (continous) $\sim$ iid

Slide 42 Assumptions:

① $\{X_i\}_{i=1}^{N}$ iid

② $N \perp \{X_i\}_{i=1}^{N}$

③ no inflation

④ $M_S(t)$, $M_N(t)$, $M_X(t)$ exist Slide 43