Week 1

1. Risk distributions / loss distribution

Review: PDF & Expectation

Continuous RV. X $\quad f_X$

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

Expectation: prob weighted average

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

n^{th} moment: $E(X^n)$

n^{th} central moment: $E\left[\left[X - E(X)\right]^n\right]$.

2. MGFs

$$M_X(t) = E(e^{tX}) = \int_a^b e^{tx} \cdot f_X(x)\,dx$$

Eg: MGF of the Uniform $(0, 1)$ r.v.

Q: If Y is a Uniform $(0, 1)$ r.v, find its MGF

Σ

Hint: $Y \sim$ Uniform(a, b). its PDF $f_Y(y) = \frac{1}{b-a}$.

$$E(e^{tY}) = \int_a^b e^{ty} \cdot f_Y(y)\,dy.$$

More generally, $E[g(x)] = \int_a^b g(x) \cdot f_Y(y)\,dy$

A: $M_Y(t) = E(e^{tY}) = \int_0^1 e^{ty} \cdot f_Y(y)\,dy$

$$= \int_0^1 e^{ty} \cdot 1\,dy$$

$$= \frac{e^t - 1}{t}$$

Note that we have $M_Y(0) = E[e^{0 \cdot Y}] = 1$

So $M_Y(t)$ is well-defined for all $t \in \mathbb{R}$

Δ Finding moments from the MGF

— use Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$ for all $x \in \mathbb{R}$

$$E\, e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} = \sum_{k=0}^{\infty} \frac{X^k t^k}{k!}$$

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} \underbrace{E(X^k)}_{k\text{th moment of } X} \frac{t^k}{k!}$$

kth moment of X is the coefficient of $\boxed{\frac{t^k}{k!}}$ in the Taylor series of $M_X(t)$

E.g. Finding the kth moment of the Uniform $(0,1)$ R.V. from its MGF.

Q: If $Y \sim$ Uniform $(0, 1)$, find $E[Y^k]$ using $M_Y(t)$.

A: From the previous e.g. we have

$$M_Y(t) = \frac{e^t - 1}{t} = \frac{1}{t}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} - 1\right)$$

$$= \frac{1}{t}\left(\sum_{k=1}^{\infty} \frac{t^k}{k!}\right)$$

$$= \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} \qquad \frac{t^{1-1}}{1!} + \frac{t^{2-1}}{2!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \boxed{\frac{t^k}{k!}} \qquad \frac{1}{0+1} \cdot \frac{t^0}{0!} + \frac{1}{1+1} \frac{t^1}{1!} + \dots$$

So the coeff of $\frac{t^k}{k!}$ for $M_Y(t)$ is $\frac{1}{k+1}$

$$\therefore E(Y^k) = \frac{1}{1+k}$$

Δ Generating moments by taking the kth derivative of $M_X(t)$

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} E(X^k) \frac{t^k}{k!}$$

$$E(X^k) = \frac{d^k}{dt^k} M_X(t) \Big|_{t=0}$$

2. Statistical distributions

2.1 Exponential distribution

$X \sim Exp(\lambda)$

Q: Find the MGF of X, $M_X(t)$, and all of its moments $E(X^k)$

A: $M_X(t) = E(e^{tX})$

$$= \cancel{e^{tx}\int_0^\infty e^{x}\lambda e^{-\lambda x}dx} \int_0^\infty e^{tx} \overbrace{\lambda e^{-\lambda x}}^{f_X(x)} dx$$

$$= \left[\frac{-\lambda}{\lambda-t} e^{-(\lambda-t)x}\right]_0^\infty, \quad t<\lambda$$

$$= \frac{\lambda}{\lambda-t}, \quad t<\lambda$$

$$= \frac{1}{1-\frac{t}{\lambda}}, \quad t<\lambda$$

Hint: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, if $|x|<1$

$$= \sum_{k=0}^{\infty} \left(\frac{t}{\lambda}\right)^k, \quad \left|\frac{t}{\lambda}\right|<1$$

$$= \sum_{k=0}^{\infty} \frac{k!}{\lambda^k} \cdot \boxed{\frac{t^k}{k!}}, \quad \left|\frac{t}{\lambda}\right|<1$$

So $E(X^k) = \frac{k!}{\lambda^k}$, for $k = 0, 1, 2 \cdots$

$\downarrow$

mean, var

$k=1$ $k=1,2$

2.2 Gamma distribution

$X \sim Gamma(\alpha, \lambda)$

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x>0$$

Δ Relationship between Gamma & χ^2

If $X \sim Gamma(\alpha, \lambda)$, and 2α is an integer, then

$2\lambda X \sim \chi^2_{2\alpha}$

If $Y \sim \chi^2_{2\alpha}$, then

$$\boxed{M_Y(t) = (1-2t)^{-\alpha}} \quad \text{for } t < \frac{1}{2}$$

Q: $X \sim$ Gamma$(10, 4)$.

Prove $Y = 8X \sim \chi^2_{20}$ distribution

Hence find approximately the Prob

the $X > 4.375$

A: The MGF of a Gamma$(10, 4)$ is

$$\left(1 - \frac{t}{4}\right)^{-10}.$$

So the MGF of $Y = 8X$ is:

$$M_Y(t) = E(e^{tY}) = E(e^{8tX})$$

$$= M_X(8t) = (1-2t)^{-10}$$

This is the MGF of an χ^2_{20} distribution

Using the uniqueness property of MGFs, Y has χ^2_{20} distribution.

$$P(X > 4.375)$$

$$= P(8X > 8 \times 4.375)$$

$$= P(Y > 35) = P(\chi^2_{20} > 35)$$

Check table $1 - 0.9799 = 0.0201$.

3. Estimation

3.1 Method of moments

A to the Eg Slide 37

(a) Step 1:

$$\text{Sample mean} = \left(200 \times \frac{2}{100} + 600 \times \frac{24}{100} + \cdots + 3400 \times \frac{1}{100}\right)$$

$$= 1216$$

Sample Variance

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$$= \left(200^2 \times \frac{2}{100} + 600^2 \times \frac{24}{100} + \cdots + 3400^2 \times \frac{1}{100}\right)$$

$$- 1216^2$$

$$= 362944$$

Step 2:

Lognormal (μ, σ^2)

Population mean $= e^{\mu + \frac{1}{2}\sigma^2}$

Population variance $= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

Step 3:

$$\boxed{\begin{array}{l}\text{Population mean} = \text{Sample mean} \\ \text{Population variance} = \text{Sample variance}\end{array}}$$

$$\boxed{e^{\mu + \frac{1}{2}\sigma^2} = 1216}$$

$$e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = 362944$$

$\sigma = 0.469$, $\mu = 6.993$

(b) $X \sim \text{LogNormal}(6.993, 0.469^2)$

$\ln X \sim \text{Normal}(6.993, 0.469^2)$

$P(X > 4000)$

$$= P\left(\underbrace{\boxed{\frac{\ln X - 6.993}{0.469}}}_{Z \sim N(0,1)} > \frac{\ln 4000 - 6.993}{0.469}\right)$$

$$= P(Z > 2.774) = 0.0028$$

3.2 Maximum likelihood Estimate (MLE)

MLE is the one that yields the highest PDF,

i.e. that max the likelihood function

A of E.g 5.50

Step 1: Write down the likelihood func

$$L = \prod_{i=1}^{n} \underbrace{\lambda e^{-\lambda x_i}}_{\text{PDF of } X \sim Exp(\lambda)}$$

$$= \lambda^n e^{-\lambda \sum x_i} \qquad \sum x_i = n\bar{x}$$

$$= \lambda^n e^{-\lambda n \bar{x}}$$

Step 2: Take natural log.

$\max L \Leftrightarrow \max \ln L$

Purpose: easy calculation

$$\ln L = n \ln(\lambda) - \lambda n \bar{x}$$

Step 3: Max log-likelihood func and solve the equations

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - n\bar{x} = 0$$

$$\hat{\lambda} = \frac{1}{\bar{x}} = \frac{1}{2200}$$

Step 4: Take second derivatives

$$\frac{\partial^2}{\partial \lambda^2} \ln L = -\frac{n}{\lambda^2} < 0$$

$\Rightarrow$ this is a max

3.3 Method of percentiles

A to E.g Slide 60

Hint: If $X \sim \text{Weibull}(c, \gamma)$,

then its CDF is $F(x) = 1 - e^{-c x^{\gamma}}$

$$\begin{cases} F(401) = 1 - e^{-c \times 401^{\gamma}} = 0.25 \\ F(2836.75) = 1 - e^{-c \times 2836.75^{\gamma}} = 0.75 \end{cases}$$

$$\Rightarrow \begin{cases} -c \times 401^{\gamma} = \ln(0.75) \\ -\cancel{c} \times 2836.75^{\gamma} = \ln(0.25) \end{cases}$$

$\Rightarrow \hat{\gamma} = 0.8038, \ \hat{c} = 0.002326$