REMEMBER from last week.

$f'(x_0)$ exists $\iff$ there exists m, B

such $f(x) = m(x-x_0) + B + r_f(x)$

where $r_f(x) = o(x-x_0)$ i.e. $\lim_{x \to x_0} \frac{r_f(x)}{(x-x_0)} = 0.$

'limiting order $o(\ )$'

Properties of differentiation

Theorem 2.1.5

Suppose I & J are open intervals in $\mathbb{R}$ and $f: I \to J$ and $g: J \to \mathbb{R}$ be functions. Let f be differentiable at $c \in I$ and g be differentiable at $f(c) \in J$. Then

$$(g \circ f)'(c) = g'(f(c)) \times f'(c)$$

where

$$(g \circ f)(c) \stackrel{\text{def}}{=} g(f(c))$$

denote $f(c) = d$ and $f(x) = y$

$$(g \circ f)(c) = g(f(c)) = g(d)$$

Proof Note: $(g \circ f)(x) - (g \circ f)(c) = g(f(x)) - g(f(c))$

$= g(y) - g(d)$ ($\Rightarrow$ by previous lemma 2.1.4)

$= g'(d)(y-d) + r_g(y) = g'(d)(f(x) - f(c)) + r_g(y)$

$= g'(d)(f'(c)(x-c) + r_f(x)) + r_g(y)$ "Prop of $r_f(x), r_g(\ )$"

$= g'(d) f'(c)(x-c) + g'(d) r_f(x) + r_g(y)$

$$\boxed{\therefore \frac{(g \circ f)(x) - (g \circ f)(c)}{(x-c)} \quad g'(d) f'(c) + g'(d) \frac{r_f(x)}{(x-c)} + \frac{r_g(y)}{(x-c)}}$$

We now need to consider the limit $x \to c$ to obtain an expression for $g \circ f$:

But we know:

1. $\lim_{x\to c} \frac{r_f(x)}{x-c} = 0$, also -

2. $\lim_{x\to c} \frac{r_g(y)}{x-c} = \lim_{x\to c} \frac{r_g(y)}{y-d} \cdot \frac{y-d}{x-c}$

Continuity of f

$$x \quad y=f(x)$$
$$\downarrow \quad \Rightarrow \quad \downarrow$$
$$c \quad d=f(c)$$

$$= \lim_{\substack{x\to c \\ \Rightarrow y\to d}} \underbrace{\left(\frac{r_g(y)}{y-d}\right)}_{\to 0} \cdot \underbrace{\left(\frac{f(x)-f(c)}{x-c}\right)}_{\to f'(c).}$$

$$= 0 \cdot f'(c) = 0.$$

2.1.4 for g at d.

$$\boxed{\therefore (g\circ f)'(c) = \lim_{x\to c} \frac{g\circ f(x) - g\circ f(c)}{(x-c)} = g'(f(c))f'(c)}$$

Theorem 2.1.6 Let $f, g : D \to \mathbb{R}$ be differential functions at $x \in D$. Let $c \in \mathbb{R}$, then cf, $f \cdot g$, $f+g$ and f/g are differentiable at x (for f/g, we require $g(x) \neq 0$), and

i) $(cf)'(x) = cf'(x)$

ii) $(f+g)'(x) = f'(x) + g'(x)$

iii) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) g'(x)$

iv) $(f/g)'(x) = \dfrac{f'(x) g(x) - f(x) g'(x)}{(g(x))^2}$

Def^{n}s

$(c \cdot f)(x) = c \cdot f(x)$

$(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$(f/g)(x) = f(x)/g(x)$

(i) $$\lim_{w\to x}\frac{(cf)(w)-(cf)(x)}{w-x}=\lim_{w\to x}\frac{c\cdot f(w)-c\cdot f(x)}{w-x}$$

$$=\lim_{w\to x}\frac{c\cdot(f(w)-f(x))}{w-x}=c\cdot\lim_{w\to x}\left(\frac{f(w)-f(x)}{w-x}\right)=c\cdot f'(x)$$

$$\therefore\ (cf)'(x)=c\cdot f'(x)$$

(ii) $$(f\pm g)(w)-(f\pm g)(x)=f(w)\pm g(w)-f(x)\mp g(x)$$
$$=f(w)-f(x)\pm g(w)\mp g(x)$$

$$\Rightarrow\lim_{w\to x}\frac{(f\pm g)(w)-(f\pm g)(x)}{w-x}=\lim_{w\to x}\frac{f(w)-f(x)}{w-x}\pm\lim_{w\to x}\frac{g(w)-g(x)}{w-x}$$

$$\Rightarrow (f\pm g)'(x)=f'(x)\pm g'(x)$$

Note: $g \circ f(x) = g(f(x))$ and $f \cdot g(x) = f(x) \cdot g(x)$

$\circ$ - composition
$\bullet$ - product

(iii) $(f\cdot g)(w) - (f\cdot g)x = f(w)\cdot g(w) - f(x)\cdot g(x)$

$$= f(w)g(w) - f(x)g(w) + f(x)g(w) - f(x)g(x)$$

$$= g(w)\left(f(w) - f(x)\right) + f(x)\left(g(w) - g(x)\right)$$

$$= g(w)\left(f'(x)(w-x) + r_f(x)\right) + f(x)\left(g'(x)(w-x) + r_g(x)\right)$$

$$\therefore \lim_{w\to x} \frac{(f\cdot g)(w) - (f\cdot g)(x)}{w-x}$$

$$= \lim_{w\to x}\left(\underbrace{g(w)}_{\to g(x)}\left(f'(x) + \underbrace{\frac{r_f(x)}{w-x}}_{\to 0}\right)\right) + \lim_{w\to x}\left(f(x)g'(x) + \underbrace{\frac{r_g(x)}{w-x}}_{\to 0}\right)$$

Note $\lim_{w \to x} \frac{\Gamma_f(w)}{w - x} = \lim_{w \to x} \frac{\Gamma_g(x)}{w - x} = 0$ (Lemma 2.1.4) $\times 2$

Therefore

$$\lim_{w \to x} \frac{(f\cdot g)(w) - (f\cdot g)(x)}{w - x} = \lim_{w \to x} f'(x)\cdot g(w) + f(x)\cdot g'(x)$$

g continuous $\to g(x)$

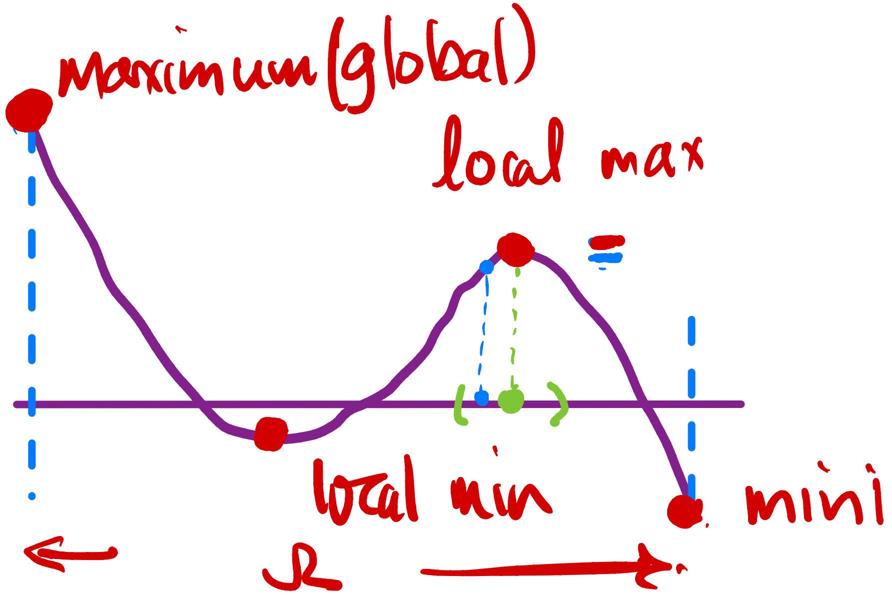
$$\Rightarrow \quad (f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

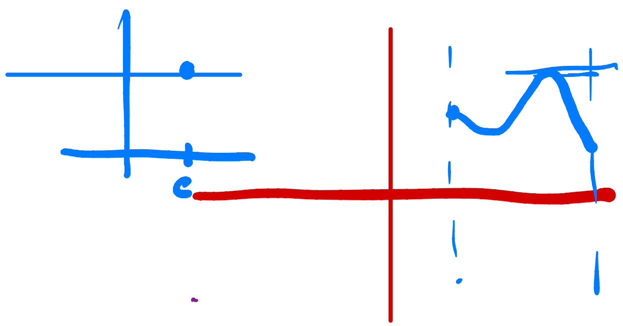
§3. Mean value theorem

A function $f:\Omega\to\mathbb{R}$ has a **local maximum** at $c\in\Omega$ if $\exists$ an open interval $I\subset\Omega$ with $c\in I$ and $f(c)\geq f(x)$, $\forall x\in I$. Also $x=c$ is a

- **strict maximum** if $f(c)>f(x)$.
- **local minimum** with $f(c)\leq f(x)$
- **strict minimum** if $f(c)<f(x)$

$\forall x\in I$





Theorem 3.1.1 Suppose $f:(a,b)\to\mathbb{R}$ is a function and suppose f has a local extremum at $c\in(a,b)$. If $f'(c)$ exists then $f'(c)=0$.

(Assume the case of local maximum)

Assume $x=c$ is a local maximum for f on an interval $I=(c-\varepsilon, c+\varepsilon)\subseteq(a,b)$

Define $q(x)=\begin{cases}\dfrac{(f(x)-f(c))}{(x-c)} & x\neq c\\ f'(c) & x=c\end{cases}\quad x\in I$

Note for $q(x)$ $\quad x<c \quad f(x)-f(c)\le 0,\ x-c<0 \quad q(x)\ge 0$

$\quad x>c \quad f(x)-f(c)\le 0,\ x-c>0 \quad q(x)\le 0$

For $x < c \quad q(x) \geq 0$
$x > c \quad q(x) \leq 0$ } + continuity of f

Prop 2.1.2
diff $\Rightarrow$ cont

$$\Rightarrow q(c) = \boxed{0 = f'(c)}$$

$$\lim_{x \to c} q(x) = \frac{f(x) - f(c)}{x - c} \quad f'(c)$$

(Note $\lim_{x \to c} q(x) = \lim_{x \to 0} \frac{f(x) - f(c)}{x - c} = f'(c) = 0$)

$\|$ continuous

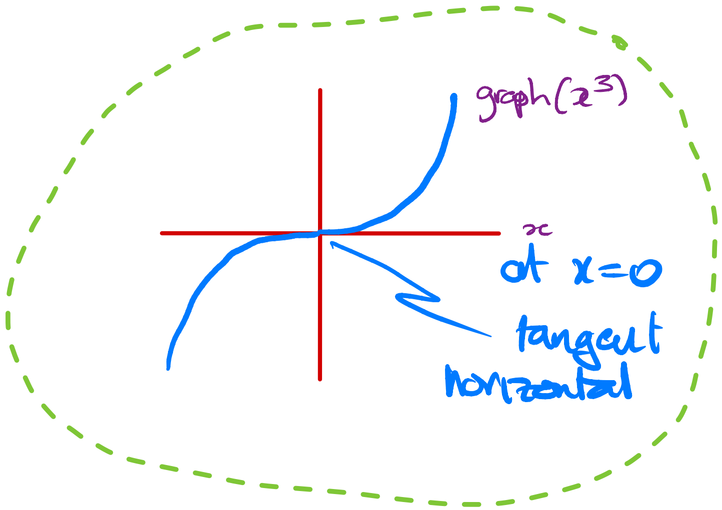
$q(c) = 0$

Note the converse is not true.

if $f(x) = x^3$, $f'(x) = 3x^2$ is zero for $x=0$,

but $x = 0$ is neither a maximum nor minimum

but a point of inflexion



Further note:

Converse of $P \Rightarrow Q$

is $Q \Rightarrow P$?

Which may or not be true!

Different from

Contrapositive

$P \Rightarrow Q \Longleftrightarrow \neg Q \Rightarrow \neg P$.

Ex1

$P =$ sum of two odd integers

$Q =$ even integer

$P \Rightarrow Q$

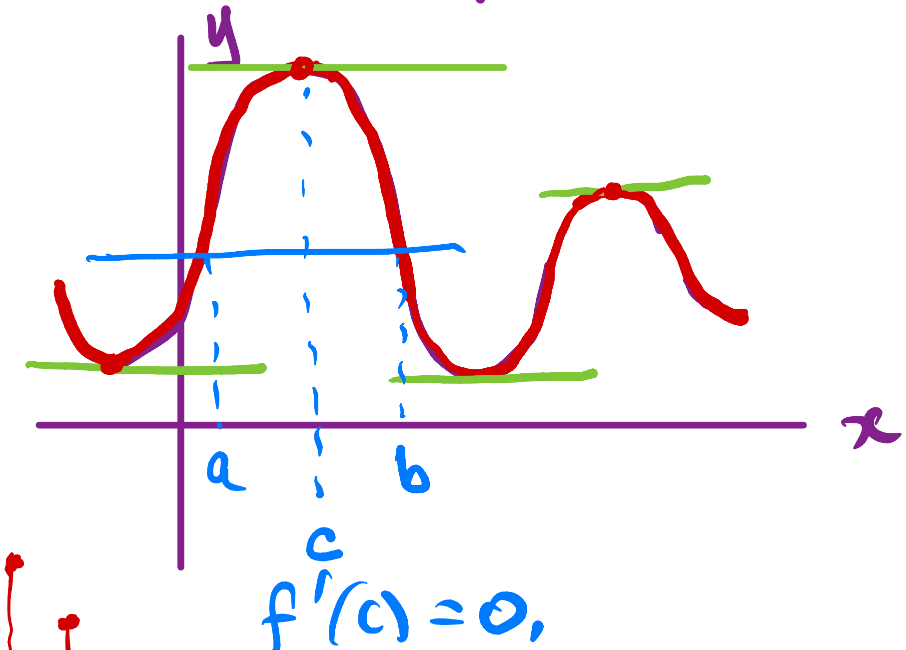
$Q \not\Rightarrow P$ counterexample $4 = 2 + 2$

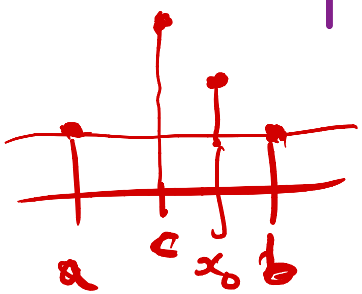
Converse is not true!

Theorem 3.1.2 (Rollé's Theorem)

Suppose $f: [a,b] \Rightarrow \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b).

If $f(a) = f(b)$, then $\exists\ c \in [a,b]$ such that $f'(c) = 0$.





Proof If $f(x) = f(a)$ $\forall x \in [a,b]$, then $f = \text{const}$ $f(x) \equiv f(a)$

$\therefore f'(x) = 0$ & theorem true

So assume $\exists x_0 \in (a,b)$

$f(x_0) > f(a) = f(b)$.

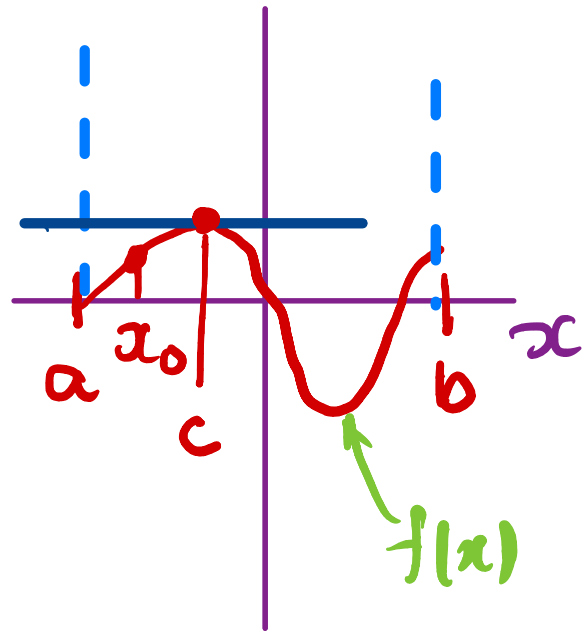
Note the maximum of f on (a,b) will be $\geq f(x_0) > f(a) = f(b)$

Now f is a continuous function on $[a,b]$

By the boundedness theorem (1.1.3) f attains its maximum at some $c \in (a,b)$

Since $f(c) > f(a) = f(b)$, $c \neq a, b$,

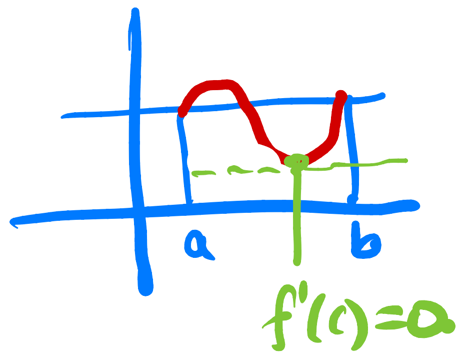
i.e By theorem 3.1.1, $f'(c) = 0$.

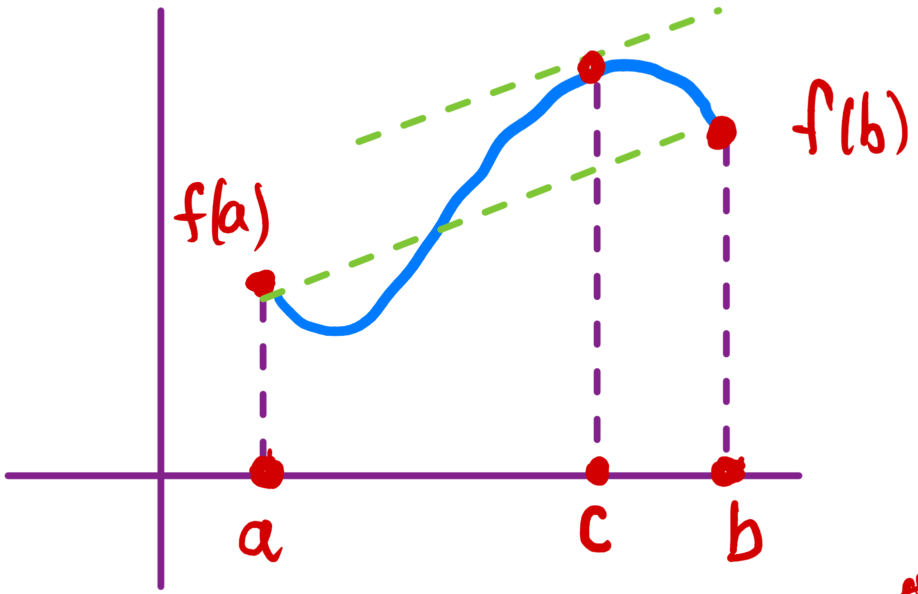


Theorem 3.1.3 (Mean Value Theorem for Derivatives)

Suppose $f:[a,b]\to\mathbb{R}$ is continous on $[a,b]$, differentiable (a,b), then $\exists\ c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$





Note: Slope of the line $f(a)$ to $f(b)$, i.e.

$\dfrac{f(b)-f(a)}{b-a}$ is the slope of the line joining $(a,f(a))$ to $(b,f(b))$

occurs at some point $c\in(a,b)$

Proof Let $\phi(x) = f(x) - f(a) - \frac{(f(b)-f(a))}{(b-a)}(x-a)$

Then

$\phi(a) = \phi(b) = 0$ (substitution)

$$\phi'(x) = f'(x) - \frac{(f(b)-f(a))}{(b-a)}(x-a)$$

ϕ satisfies the hypotheses of Rollé's Theorem (Theorem 3.1.2) $\therefore \exists\, c \in (a,b)$ such that $\phi'(c) = 0$.

and so $f'(c) = \frac{f(b)-f(a)}{b-a}$ (substitution in $\phi'(x)$)

Note: Rolle's Thm and the MVT are <u>EQUIVALENT</u>

each can be proven from the other.

(1) Assume MVT: Let $f(a)=f(b)$, then $\exists\ c \in (a,b)$ with $f'(c)=0$ (which is Rolle's Theorem)

(2) Assume Rolle's Theorem:

use $\varphi(x) = f(x) - f(a) - \dfrac{(f(b)-f(a))}{(b-a)}(x-a)$

and apply Rolle's Theorem

to $\exists\ c \in (a,b)$ such that $\varphi'(c)=0$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

Application of Mean Value Theorem

Thm 3.1.4 Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).

If $f'(x) = 0 \quad \forall x \in (a,b)$, then f is constant on $[a,b]$,

i.e. $f(x) \equiv c$, for some $c \in \mathbb{R}$.

Proof Let $x \in [a,b]$ and consider f on $[a,x]$

f satisfies the conditions of the MVT on $[a,b]$ and therefore also on $[a,x]$

$x \neq a$

Therefore,

$$f(x) - f(a) = f'(c_{x,a})(x-a)$$

$$f'(c_{x,a}) = 0$$

$[a, x] \subseteq [a,b]$

MV Thm.

for some $c_{x,a} \in (a,x)$.

$\therefore f(x) = f(a)$ since $f'(c_{x,a}) = 0$.

Given $x \in [a,b]$ was chosen arbitrarily,

$f(x) = f(a)$, $\forall x \in [a,b]$.

$f(x) \equiv$ const.

$\square$

Thm 3.1.5 Assume that $|f'(x)| \leq M$, $x \in [a,b]$ and f is continuous on $[a,b]$, then

$$|f(x)-f(y)| \leq M|x-y|$$

Notation $c_{(x,y)}$ indicates some $c \in (x,y)$

Proof WLOG assume $y > x$,

Apply MVT on $[x,y]$, and we get

$$\underline{|f(x)-f(y)|} = \underline{|f'(c_{(x,y)})|}\,\underline{|(x-y)|}$$

for some $c_{(x,y)} \in (x,y)$

$$\therefore \quad |f(x)-f(y)| \leq |f'(c_{(x,y)})|\,|x-y| \leq M|x-y|$$

This gives a bound on $|f(x)-f(y)|$

Thm 3.1.6 Assume $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and diff on (a,b).

non-decreasing

$x < y \Rightarrow f(x) \leq f(y)$

Then, for $\forall\, x \in (a,b)$,

(i) $\boxed{f' \geq 0, \quad f \text{ is non-decreasing}}$ $\Rightarrow$ $\Leftarrow$

(ii) $\boxed{f' \leq 0, \quad f \text{ is non-increasing}}$ $\Rightarrow$ $\Leftarrow$

(iii) $\boxed{f' > 0, \quad f \text{ is strictly increasing}}$ $\Rightarrow$ TRUE ✓ $\not\Leftarrow$ FALSE ✗

(iv) $\boxed{f' < 0, \quad f \text{ is strictly decreasing}}$ $\Rightarrow$ TRUE ✓ $\not\Leftarrow$ FALSE ✗

CONVERSES for Thm 3.1.6 ?

Be careful with converses (iii) converse not true

Note $f(x)=x^3$, f is strictly increasing but f' is only $f' \geq 0$

WHY?

let $x<y$: $y^3 - x^3 = (y-x)(x^2+xy+y^2)$

- $0 \leq x < y$: then $y-x>0$ and $x^2+xy+y^2>0$, $\therefore y^3 > x^3$

- $x<y<0$: then $y-x>0$ and $x^2+xy+y^2>0$ $\therefore y^3>x^3$

(× of 2 negatives is positive)

- $x \leq 0 \leq y$: $x \neq y$, $x^3 \leq 0$, $y^3 \geq 0$

$\Rightarrow x^3 < y^3$ if either x or $y \neq 0$

Proof

We prove (i) and leave (ii), (iii), (iv) as exercises.

Let $x, y \in (a,b)$ and assume $x < y$.

By the MVT on $[x,y]$,

$$f(y) - f(x) = f'(c)(y-x),$$

for some $c \in (x,y)$.

Now $y > x$ & $f'(c) \geq 0 \Rightarrow f(y) \geq f(x)$ for $y \geq x$

The inverse to (iii) & (iv) is not true!

e.g. $f(x) = x^3$, $x \in [-1,1]$ satisfies $f'(x) \geq 0$ $(f'(0) = 0)$

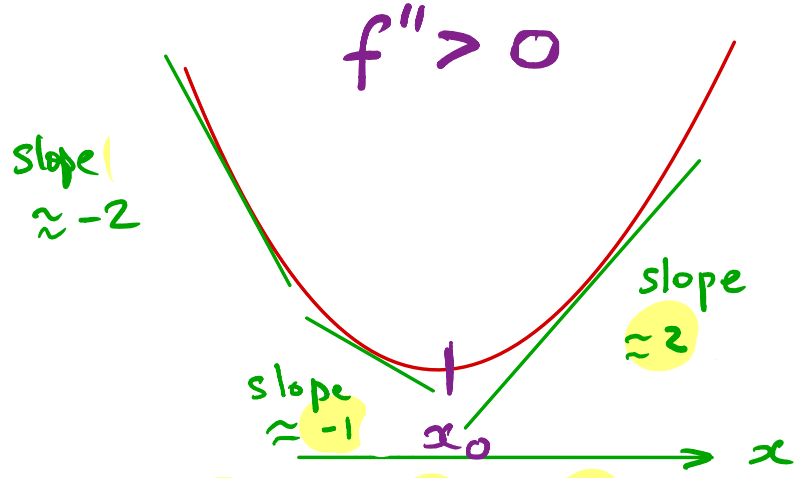
but $y > x \Rightarrow f(y) > f(x)$.

MINIMUM $f'' > 0$

f & f', $\boxed{f' > 0 \Rightarrow f \text{ increases}}$

Analogous situation for f' & f''

$$\boxed{f'' > 0 \quad f' \text{ increases through } x_0 \text{ where } f'(x_0) = 0}$$



SLOPE $-3 \rightarrow -1 \rightarrow 2$

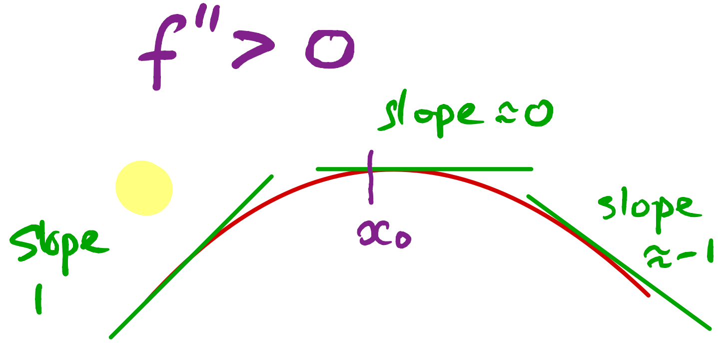
f' increases as x increases

MAXIMUM $f'' < 0$

f & f', $\boxed{f' < 0,\ f \text{ decreases}}$

Analogous situation for f' & f''

$$\boxed{f'' < 0 \quad f' \text{ decreases through } x_0 \text{ where } f'(x_0) = 0}$$



SLOPE $1 \rightarrow 0 \rightarrow -1$

f' decreases as x increases

Theorem 3.1.7 Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$.

Assume f is twice diff on (a,b). Let $x_0 \in (a,b)$, then

(i) $f'(x_0)=0$, $f''(x_0)>0 \Rightarrow x_0$ is a local minimum

(ii) $f'(x_0)=0$, $f''(x_0)<0 \Rightarrow x_0$ is a local maximum

Proof Let's prove (i), otherwise consider $-f$.

By definition

$$f''(x_0) = \lim_{x\to x_0}\frac{f'(x)-\overset{0}{f'(x_0)}}{x-x_0} = \lim_{x\to x_0}\frac{f'(x)}{x-x_0} > 0$$

why?

But f' is differentiable, therefore continuous at x_0.