Mathematical Tools For Assest Management
MTH6113

Topic 2

Utility Theory
and Expected Utility Theory

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Plan

Utility theory

- Two goods
- Two goods and Cobb-Douglas utility

Expected Utility Theory

- Fair Bet
- Risk Aversion/Seeking/Neutrality
- Certainty Equivalent of a Gamble
- Measures of Risk Aversion

Consumer's Decision Problem - two goods

Any agent's **decision problem**:

$$\max u(x, y)$$

- such that the budget constraint is satisfied:

$$p_x x + p_y y \leq m$$

- p_x- price of x
- p_y- price of y
- m- total available income

Consumer's Decision Problem - two goods

Optimisation problem with an inequality constraint: use **Lagrangian method**

- Set Lagrangian function:

$$\mathcal{L}(x, y, \lambda) = u(x, y) + \lambda(m - p_x x - p_y y)$$

- λ- Lagrange multiplier

Consumer's Decision Problem - two goods

First Order Conditions

At the optimum:

1. $$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial u\left(x, y\right)}{\partial x} - \lambda p_x = 0$$

2. $$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial u\left(x, y\right)}{\partial y} - \lambda p_y = 0$$

3. $$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_x x - p_y y = 0$$

Solution: $\left(x^*, y^*, \lambda^*\right)$

Combining 1 and 2

At the optimum:

$$\frac{\partial u(x,y)}{\partial x} / \frac{\partial u(x,y)}{\partial y} = \frac{p_x}{p_y}$$

or

$$MRS_{x,y} = \frac{p_x}{p_y}$$

Together with

$$p_x x + p_y y = m$$

Sistem of two equations, two unknowns.

Consumer's Decision Problem: Example

In R_{+}^{2} the Cobb-Douglas utility function is given by:

$$u(x_1, x_2) = x_1^a x_2^b, \text{ with } 0 < a, b \leq 1$$

The consumer's optimisation problem is:

$$\max_{x_1, x_2} u(x_1, x_2) = x_1^a x_2^b \text{ subject to}$$

$$p_1 x_1 + p_2 x_2 \leq m$$

The Langrangian function in this case is:

$$\mathcal{L}(\lambda, x_1, x_2) = x_1^a x_2^b + \lambda (m - p_1 x_1 - p_2 x_2)$$

Consumer's Decision Problem: Example

$$\begin{aligned} a x_1^{a-1} x_2^{b} - \lambda p_1 &= 0 \\ b x_1^{a} x_2^{b-1} - \lambda p_2 &= 0 \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

This system can be simplified to:

$$\begin{aligned} \frac{a x_2}{b x_1} &= \frac{p_1}{p_2} \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

Consumer's Decision Problem: Example

Solution:

$$
\begin{aligned}
x_1^*(p_1, p_2, m) &= \frac{m}{p_1}\frac{a}{a+b} \\
x_2^*(p_1, p_2, m) &= \frac{m}{p_2}\frac{b}{a+b}
\end{aligned}
$$

Consumer's Decision Problem: Example

The second order condition for a local maximum can be written in terms of Bordered Hessian:

$$\begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{pmatrix}$$

The determinant of the bordered Hessian is positive:

$$\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{vmatrix} > 0$$

As $u_{11}, u_{22} < 0$ and $u_{12} = u_{21} > 0$ for all $x_1, x_2 > 0$ is satisfied.

Expected Utility Theory

Generalise utility theory to consider situations that involve **uncertainty**

- decision over investment choices
- decision maker
- utility of wealth

Any **risky asset** is characterised by a set of objectively known probabilities defined on a set of possible outcomes

Expected Utility Theory

The expected utility of a risky asset:

$$E\left[U\left(W\right)\right] = \sum_{i=1}^{N} p_i u\left(w_i\right)$$

When uncertainty present it is impossible to maximise utility with complete certainty

Maximise the expected value of utility given investor's particular beliefs about the probability of different outcomes

Assumptions

1. *Completeness (or Comparability)*:

- either $U(\mathbf{x}) > U(\mathbf{y})$, or $U(\mathbf{y}) > U(\mathbf{x})$, or $U(\mathbf{x}) = U(\mathbf{y})$

2. *Transitivity*:

- if $U(\mathbf{x}) > U(\mathbf{y})$ and $U(\mathbf{y}) > U(\mathbf{z})$, then $U(\mathbf{x}) > U(\mathbf{z})$,

3. *Local non-satiation or More is Better*.

- $U'(\cdot) > 0$ - marginal utility of wealth is strictly positive

4. *Independence*

- If an investor is indifferent between two certain outcomes, **x** and **y**, then he is also indifferent between the gambles (or lotteries):
  - **x** with probability p and **z** with probability $(1-p)$, and
  - **y** with probability p and **z** with probability $(1-p)$.

$$pU(\mathbf{x}) + (1-p)U(\mathbf{z}) = pU(\mathbf{y}) + (1-p)U(\mathbf{z})$$

Expected Utility Theory

5. *Certainty Equivalence*

- If

$$U(\mathbf{x}) > U(\mathbf{y}) > U(\mathbf{z})$$

then there exists a unique $0 < p < 1$ such that

$$pU(\mathbf{x}) + (1-p)U(\mathbf{z}) = U(\mathbf{y})$$

y - the certain level of wealth that yields the same certain utility as the expected utility yielded by the gamble

y - loosely speaking - the maximum price that an investor would be willing to pay to accept a gamble

Uncertainty involves taking risks

What is our attitude towards risk?

- *Example:* I toss a fair coin. If it is a head, you give me £5 and if it is a tail, I give you £5 Would you accept this gamble?

$$E(w) = \frac{1}{2}(w_0 - 5) + \frac{1}{2}(w_0 + 5) = w_0$$

- If the expected wealth is equal to initial wealth (w_0) the **gamble is fair**
- However, different people have different attitudes towards risk:

$$E(U(w)) \ ? \ U(w_0)$$

Utility Theory and Risk

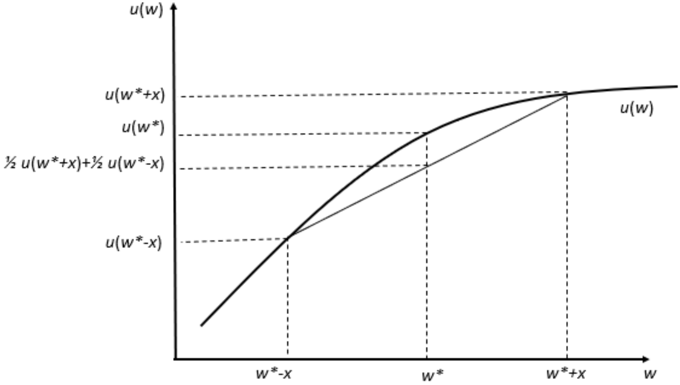
Risk aversion
A risk averse investor will reject a fair gamble

- he attaches a lower utility to an incremental increase in wealth to an incremental decrease so $U^{''}(w) < 0$

The utility function of a risk averse investor:

- is a strictly concave function of wealth
- hence, exhibits diminishing marginal utility of wealth

Risk Aversion



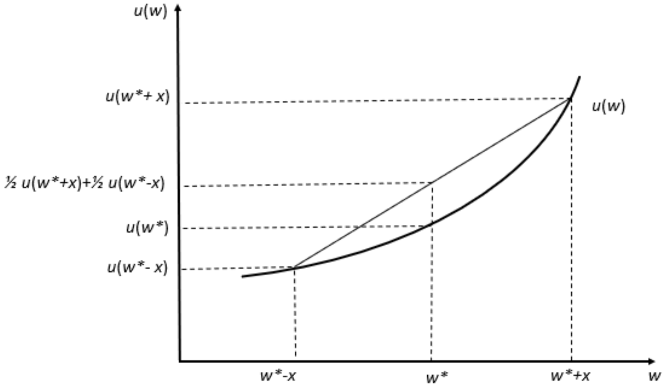
Risk seeking
A risk seeking investor will seek a fair gamble

- he attaches a higher utility to an incremental increase in wealth to an incremental decrease so $U''(w) > 0$

The utility function of a risk seeking investor:

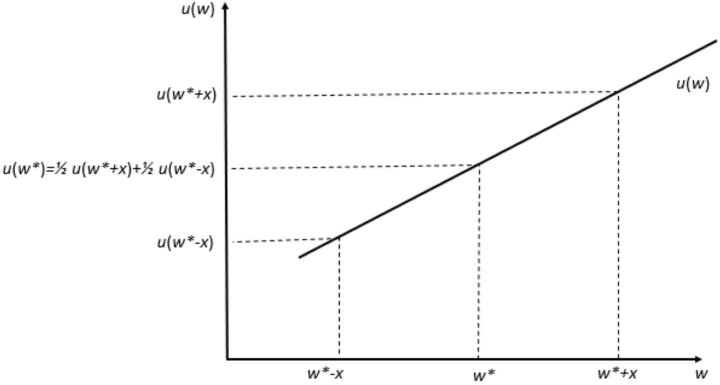
- is a strictly convex function of wealth
- hence, exhibits increasing marginal utility of wealth

Risk seeking



Risk neutrality

A risk neutral investor is indifferent to weather to accept or not a fair gamble

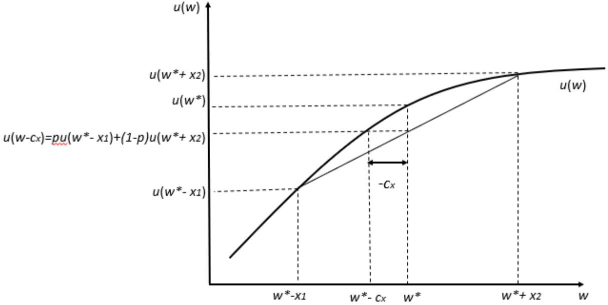


The certainty equivalent of a gamble x**,** denoted c_x is determined by

$$E\left(U\left(w+x\right)\right)=U\left(w-c_x\right)$$

Utility Theory and Risk

If that the gamble takes values: $\{-x_1, x_2\}$ with probabilities $\{p, (1-p)\}$, c_x diagramatically c_x is:



Utility Theory and Risk

If the gamble is fair then a risk averse investor will reject a fair gamble i.e. keep their current wealth

$$E\left(U\left(w+x\right)\right)=U\left(w-c_x\right)<U\left(w\right)$$

The investor pays c_x to avoid the gamble (or has to be paid to take the gamble)

The principal underlying insurance

Utility Theory and Risk

- Degree of risk aversion reflected in the degree of concavity of the utility function
- Attitude to risk may change depending on current level of wealth
  - need to take account of the initial wealth

Absolute risk aversion

- The investor exhibits decreasing (increasing) absolute risk aversion (ARA) if c_x decreases (increases) as wealth increases
  - Decreasing ARA: as wealth increases the absolute amount of wealth in risky assets increases

Relative risk aversion

- The investor exhibits decreasing (increasing) relative risk aversion (RRA) if $\frac{c_x}{w}$ decreases (increases) as wealth increases
  - Decreasing RRA: as wealth increases the relative amount of wealth in risky assets increases

Arrow-Pratt measures of Risk Aversion

Absolute Risk Aversion

$$A(w) = -\frac{U''(w)}{U'(w)}$$

Relative Risk Aversion

$$R(w) = -w\frac{U''(w)}{U'(w)}$$

Based on c_x proportional to $\frac{U''(w)}{U'(w)}$

Risk Aversion: Absolute and Relative

	ARA	CRA
Increasing	$A'(w) > 0$	$R'(w) > 0$
Decreasing	$A'(w) < 0$	$R'(w) < 0$
Constant	$A'(w) = 0$	$R'(w) = 0$