Week 2 Monday lecture

PLAN: (1) Wave equation and Lorentz boosts

(2) Special relativity

(3) Spacetime diagrams

↳ Time dilatation and length contraction

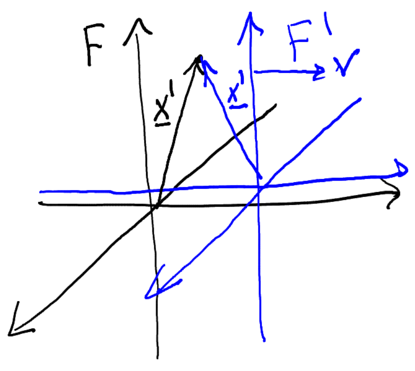
(4) Four-vector notation

(1) In the previous lectures we saw that electromagnetic fields propagate at a finite velocity (c). How should we interpret this fact?

- Option 1: there is a medium (aether) through which $\underline{E}$, $\underline{B}$ propagate and c is the velocity measured by an observer at rest with respect to the aether "fluid"
- Option 2: c is a fundamental constant of nature and is the same in all inertial frames

At the end of XIX century, there were several experimental attempts to measure the relative motion of the earth through the aether most famously thanks to Michelson and Morley. They devised an experimental apparatus (the interferometer) whose evolution is at the basis of the current experiments carried out by LIGO/VIRGO/KAGRA... so this idea certainly played a central role in the development of special and general relativity! No aether was detected so let's explore option 2. It is clear in contrast with the invariance under Galilean transformations and in fact Galilean boosts are not symmetries of wave equation (COURSEWORK).

Thus the question is whether there exists a generalisation of transformation law for the boosts that leaves the wave equation invariant



$\underline{v} = (v, 0, 0)$ relative velocity between F and F'.

Start from $\psi(t, \underline{x})$ in frame F and $\psi'(t', \underline{x}')$ assuming $\psi'(t', x') = \psi(t, x)$: what relations between $(t', \underline{x}')$ and $(t, \underline{x})$ which ensure that

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial}{\partial t'^2}\right)\psi'(t', \underline{x}') =$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right)\psi(t, \underline{x}) \quad ?$$

(2) In analogy with Galilean transformation we will assume that there is a <u>linear</u> relation between $(t', \underline{x}')$ and $(\underline{t}, \underline{x})$, so we can write

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} L^0{}_0 & L^0{}_1 \cdots & L^0{}_3 \\ L^1{}_0 & & \vdots \\ \vdots & & L^3{}_3 \\ L^3{}_0 \cdots & \cdots & \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

L is a 4x4 constant matrix which depend on V.

(I write ct instead of t just for convenience: in this way all the objects we deal with have the physical dimension of length). For later convenience I start labeling the matrix elements from 0 (instead of 1 as you did in Linear Algebra).

In the Galilean case we would have ↗ Kronecker delta

$$\left.\begin{array}{l} t' = t \\ \underline{x}' = \underline{x} - \underline{v}t \end{array}\right\} \Rightarrow \begin{array}{l} L^0{}_0 = 1, \quad L^i{}_j = \delta_{ij}, \quad i,j = 1,2,3 \\ L^0{}_i = 0, \quad \begin{pmatrix} L^1{}_0 \\ L^2{}_0 \\ L^3{}_0 \end{pmatrix} = \begin{pmatrix} -v_x/c \\ -v_y/c \\ -v_z/c \end{pmatrix} \end{array}$$

Let us take inspiration from the simplest Galilean boost $\underline{v} = (v, 0, 0)$ and assume that t and x should transform in a similar way since in the wave equation they appear almost symmetrically (both as a double derivative):

$$\boxed{\begin{array}{l} x' = \gamma(x - vt) = \gamma\left(x - \frac{v}{c}(ct)\right) \\ ct' = \gamma\left(ct - \frac{v}{c}x\right), \quad y' = y, \quad z' = z \end{array}} \Rightarrow \begin{array}{l} L^0{}_0 = L^1{}_1 = \gamma \\ L^0{}_1 = L^1{}_0 = -\gamma\frac{v}{c} \\ L^i{}_j = \delta_{ij} \end{array}$$

<u>Coursework</u>: check that the transformation above is a symmetry of the wave equation provided that

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These transformations were introduced by Lorentz.

For a light ray travelling in the x-direction, we have

$$\frac{dx'}{dt'} = \frac{d(\gamma(x-vt))}{d(\gamma(t-\frac{v}{c^2}x))} \underset{\text{const.}}{\overset{v \text{ is}}{=}} \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{c-v}{1-\frac{v}{c}} = c$$

where I used $\frac{dx}{dt} = c \Rightarrow$ so the speed of light is not changed by Lorentz boosts (as wanted!).

Comment: Notice that we must have $v<c$, otherwise γ is ill-defined $(v=c)$ or imaginary $(v>c)$.
Thus c is a limit speed that no observer can reach, while light travel at this speed (c) in all frames.

Exercise: $\gamma\left(\frac{|v|}{c}\right) \geq 1$ (plot this function in the allowed region $0 \leq |v|/c < 1$)

Exercise: if $v \ll c$ (as in "normal situations, one can approximate the Lorentz boost by sending $\frac{|v|}{c} \to 0$ but keeping v fixed: show that the law for Galilean boosts is recovered

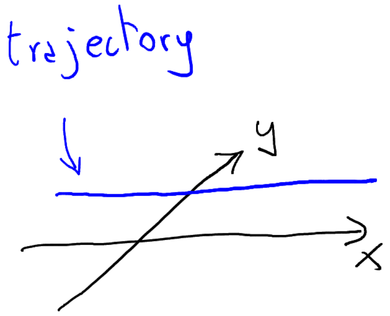
(3) A key difference between the types of boost transformations is the non-trivial mixing between space and time in the transformation of t

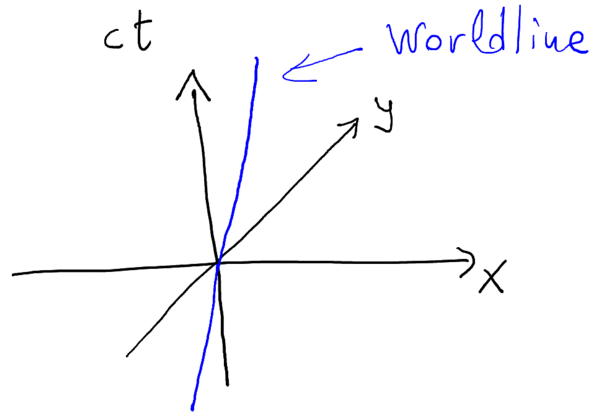
$$ct \to ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

Notice that this is possible only thanks to the existence of a universal velocity c which can be used to have the right physical dimensions (a guess along the lines $t \to t' = t - \frac{x}{|v|}$ is problematic for several reasons: think for instance at the $v \to 0$ case).

A first consequence is that we should consider frames that treat democratically space and time. So instead of depicting the trajectory of particle in physical space, we follow our particle through spacetime.

Example (in 2 spatial dimension)

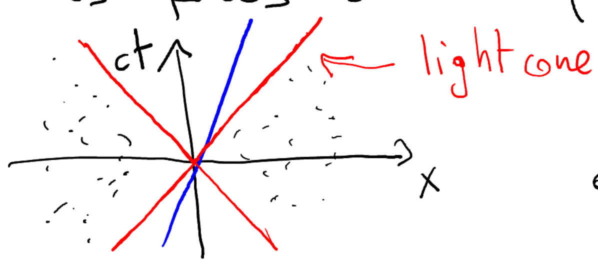




A trajectory for a particle moving along x and at rest along y

A world-line for a particle with a uniform straight motion along x (and at rest along y)

Let us focus on the (ct, x) subspace



The red lines would describe objects propagating along x

at velocity $\frac{dx(t)}{dt} = c$. No physical object passing throug $(ct, x) = (0, 0)$ can reach the dotted region as it would require a superluminal speed $(|v| > c)$ [recall that the slope of the tangent to the world line is the velocity $v = \frac{dx(t)}{dt}$].

By definition each point in spacetime identifies an <u>event</u> specifying when and where it happens.

Time dilatation: consider a clock at rest in a frame F' which moves with velocity v along x with respect to the frame F. In F' a time interval $\Delta t'$ is difference between two events

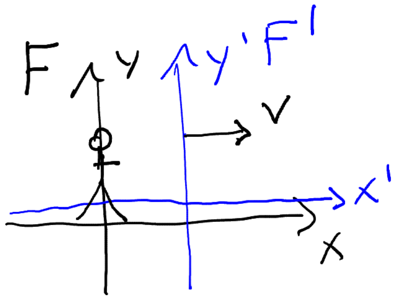
$$\Delta(ct', 0) = (ct'_2 - \underbrace{x'_c}_{\text{position of the clock}}) - (ct'_1 - x'_c)$$

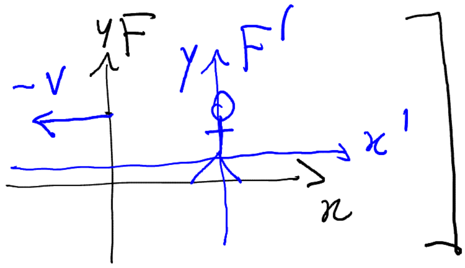
position of the clock which is at rest in the frame F'

In F these two events correspond to

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{v}{c}x\right) \\ x' &= \gamma\left(x - \frac{v}{c}(ct)\right) \end{aligned} \iff \begin{aligned} ct &= \gamma\left(ct' + \frac{v}{c}x'\right) \\ x &= \gamma\left(x' + \frac{v}{c}(ct')\right) \end{aligned}$$

[You can check this mathematically, by inverting the first set of relations or physically by using the if F' has velocity $+v$ with respect to F, then F has velocity $-v$ with respect to F'





Then the time interval measured is read from

$$(ct_2, x_c(t_2)) - (ct_1, x_c(t_1)) = \left(\gamma\left(ct_2' + \frac{v}{c}x_c'\right), x_c'\right) - \left(\gamma\left(ct_1' + \frac{v}{c}x_c'\right), x_c'\right)$$

$$\Downarrow$$

$$(c\Delta t, \#) = (\gamma c \Delta t', 0)$$

So $\Delta t = \gamma \Delta t'$ and, since $\gamma > 1$, $|\Delta t| > \gamma |\Delta t'|$

Notice that this time dilatation depends only on v^2 and not on the direction of the velocity.

<u>Length contraction</u>: consider a rod stretching from x_1' to $x_2' = x_1' + l'$ at rest in the frame F'. What length would measure an observer with a relative velocity

v. with respect to F (F and F' have the same relative boost as in the case above)?

The length is measured by looking at the end point at the same time. Of course in the F' frame time does not matter as the rod does not move, so we can measure the length by considering

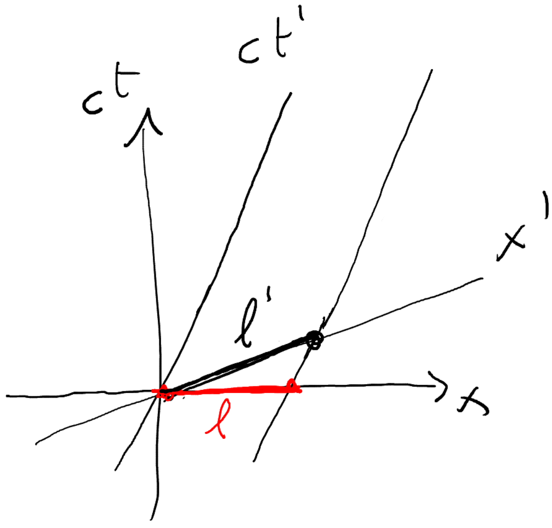
$$\ell' = x_2'(t_2') - x_1(t_1') = \gamma\left[(x_2(t_2) - v t_2) - (x_1(t_1) - v t_1)\right]$$

Thus it is convenient to choose $t'_{1,2}$ such that $t_1 = t_2$ which implies $\ell' = \gamma\underbrace{(x_2(t_1) - x_1(t_1))}_{\text{Length } \ell \text{ in the frame F}} = \gamma \ell$

Thus $\ell = \frac{1}{\gamma}\ell'$, so in the frame F the rod length is shorter ($\gamma > 1$ if $v \neq 0$) than in F'.

Comment: add another spatial direction (y). If the rod is aligned to y (while the boost between F and F' is still along x) it is immediate that there is no contraction as $y' = y$

Geometrically we have



(for $v>0$)

The t' axis is the trajectory of the $x'=0$ point, i.e it corresponds to the line $x=\frac{v}{c}(ct)$ [from $x'=\gamma(x-vt)$]

(4) Since ct and the spatial coordinates mix under Lorentz boost let us think at spacetime as a single 4D space

$$(ct, x, y, z) \longleftrightarrow (x^0, x^1, x^2, x^3) \longleftrightarrow x^a$$

where the spacetime index run from 0 to 3.

I will use roman indices from the middle of the alphabet to indicate just the space directions

$$(x, y, z) \longleftrightarrow (x^1, x^2, x^3) \longleftrightarrow x^i$$

Thursday lecture

Postponed

Plan: (5) The Lorentz and the Poincaré groups

(6) Upper/lower indices (vectors/1-forms)

(5) It is convenient to parametrise the boost velocity in terms of the rapidity β

$$\frac{1}{2}\left(e^{\beta}+e^{-\beta}\right) \equiv \cosh\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma$$

Exercise: Use the properties of the exp-function to show $\cosh^2\beta - \sinh^2\beta = 1$ and $\sinh\beta = \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}}$

Thus we can write the Lorentz boost between F' and F above as

$$x'^0 = \cosh\beta\, x^0 - \sinh\beta\, x^1$$
$$x'^1 = -\sinh\beta\, x^0 + \cosh\beta\, x^1$$

or in matrix notation

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

This is the matrix L on pag. 3 for the boost between F' and F. In component

$$x'^a = \sum_{b=0}^{3} L^a{}_b \, x^b$$

When indices are repeated as in the r.h.s. we will assume that a sum over all possible values of the repeated index should be included. So we'll write the relation above as $x'^a = L^a{}_b \, x^b$.

We are now in the position of giving a simple general definition of the Lorentz transformations: they are represented by constant matrices L that preserve (i.e. leave unchanged) the "relativistic length" of a

NOTICE THE MINUS !!

$$x^\mu = (x^0, x^1, x^2, x^3) \quad \text{and} \quad v^2 \equiv -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

Let us then introduce a key object (η_{ab}), the <u>Minkowski</u> <u>metric</u>

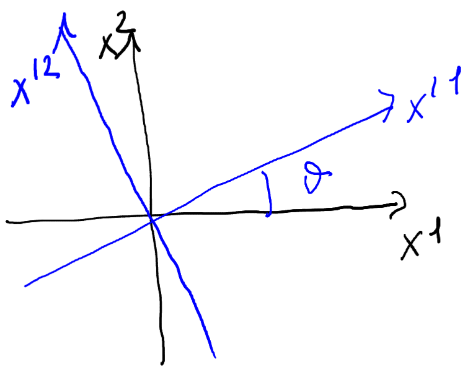
$$\eta_{ab} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{22} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then we have $x^2 = x^a \eta_{ab} x^b$ (with the summation convention understood). Notice that η_{ab} is a symmetric, non-singular matrix.

- An easy class of transformations L with this feature are of the form
$$L^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$
i.e.

they do not change the time component and act non-trivially only on the space coordinates. They are part of the Galilean transformations as well
$$x'^i = R^i{}_j \, x^j \quad \text{with} \quad (R^T)_i{}^k \, \delta_{kl} \, R^l{}_j = \delta_{ij}$$
or $R^T R = I_{3\times3}$ in matrix notation. They represent a rotation (or a reflection) in space and form the group of 3x3 orthogonal matrices $O(3)$.

Example: rotation in the (x^1, x^2) plane



$$R = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The transformations involving time are Lorentz boost

Starting from the example on pag. 11 and focusing on the directions (x^0, x^1), where the transformation acts non-trivially, we have to check that

$$x'^2 = (x')^a \eta_{ab} (x')^b = (L^a{}_c x^c) \, \eta_{ab} \, (L^b{}_d x^d) =$$

$$x^c \left[(L^T)_c{}^a \eta_{ab} L^b{}_d \right] x^d \stackrel{?}{=} x^c \eta_{cd} x^d = x^2$$

So the relativistic length is invariant iff

$$L^T \eta L = \eta \iff (L^T)_a{}^c \eta_{cd} L^d{}_b = \eta_{ab}$$

$$\underbrace{\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}}_{L^T} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\eta} \underbrace{\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}}_{L} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\eta}$$

thanks to $\cosh^2\beta - \sinh^2\beta = 1$

This is called Lorentz group

The set of all transformations preserving η_{ab} forms the group $O(1,3)$: it is similar to $O(4)$ except for the presence of the minus sign in η. As already mentioned, this set of transformations is <u>not</u> part of the Galilean group.

• Finally there are the space and the time shifts

$$x^a \to x'^a = x^a + d^a$$ where $d^a = (d^0, d^1, d^2, d^3)$ is

a constant vector. They keep the "relativistic length" unchanged and so are relevant for the relativistic case as well. Combining Lorentz transformations with the spacetime shifts, one generates a larger group of symmetries called Poincaré group.

Comment: The number of continuous independendent generators of the Poincaré group is 10 ... exactly the same as the case of Galilean transformations.

6) As done in the equations above we will use upper indices to indicate the components of a vector (x^a). We can define objects with lower indices by using the Minkowski metric

$$x_a \equiv \eta_{ab} x^b$$ summed

Notice that $x_0 \neq x^0$ as

$$(x_0, x_1, x_2, x_3) = (-x^0, x^1, x^2, x^3)$$

Then in order to raise the indices we need a matrix $\hat{\eta}^{ca}$ which is the inverse of η_{ab}, i.e.

$$\hat{\eta}^{ca}\,\eta_{ab} = \delta^c_{\ b} \leftarrow \text{identity matrix so}$$

using

$$\hat{\eta}^{ca}\,x_a = \underbrace{\hat{\eta}^{ca}\,\eta_{ab}}_{\text{summed}}\,x^b = \delta^c_{\ b}\,x^b = x^c$$

But since η is its own inverse (as a 4x4 matrix) we can drop the hat and use η^{ca}.

Objects with an upper index are the coordinate of a vector, while objects with a lower index are the coordinates of a co-vector (also known as a 1-form).

<u>Exercise</u> : Start from the transformation of x^a under a boost of velocity v along x^1. Show that x_a transforms by using the <u>inverse</u> boost (i.e. a boost of velocity $-v$ along x_1).

Hint: use $(x_0, x_1, x_2, x_3) = (-x^0, x^1, x^2, x^3)$