MTH5113 (2023/24): Problem Sheet 2

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 1**.

---

(1) *(Warm-up)* Compute each of the following:

(a) Consider the vector-valued function

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{f}(t) = (t^2, t^3 - 1).$$

(i) Compute $\mathbf{f}'(t)$ for every $t \in \mathbb{R}$.

(ii) Find the values $\mathbf{f}'(0)$, $\mathbf{f}'(1)$, and $\mathbf{f}'(-2)$.

(b) Consider the vector-valued function

$$\mathbf{g} : (0, 1) \to \mathbb{R}^3, \qquad \mathbf{g}(t) = (\ln t, \ln(1 - t), e^{3t} + t).$$

(i) What happens to $\mathbf{g}(t)$ as t approaches 0? As t approaches 1?

(ii) Compute $\mathbf{g}'(t)$ for every $t \in \mathbb{R}$.

(iii) Compute the second derivative $\mathbf{g}''(t)$ for every $t \in \mathbb{R}$.

(2) *(Warm-up)* Let $\mathbf{A}$ denote the vector-valued function

$$\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^2, \qquad \mathbf{A}(x, y, z) = (x(1 - z), y(1 - z)).$$

(a) Compute the partial derivatives $\partial_1 \mathbf{A}(x, y, z)$, $\partial_2 \mathbf{A}(x, y, z)$, and $\partial_3 \mathbf{A}(x, y, z)$ at every point $(x, y, z) \in \mathbb{R}^3$.

(b) Find $\partial_2 \mathbf{A}(1, 0, 3)$ and $\partial_3 \mathbf{A}(0, -1, -1)$.

(3) *(Warm-up)* Let $\mathbf{F}$ be the vector field on $\mathbb{R}^2$ defined via the formula

$$\mathbf{F}(x, y) = (x - y, x + y)_{(x,y)}.$$

(a) Compute the following: (i) $\mathbf{F}(1,-1)$; (ii) $\mathbf{F}(-2,-1)$; (iii) $\mathbf{F}(-1,\frac{1}{2})$.

(b) Plot the three tangent vectors from part (a) onto a Cartesian plane.

(4) [**Tutorial**] Consider the following vector-valued function:

$$\mathbf{h} : (0,\infty) \to \mathbb{R}^2, \qquad \mathbf{h}(t) = (t\cos t,\ t\sin t).$$

(a) Sketch the values $\mathbf{h}(t)$, for all $0 < t < 4\pi$. Also, plot the values of $\mathbf{h}$ on computer (see the *Additional Resources* section on the *QMPlus* page).

(b) Compute $\mathbf{h}(\pi)$ and $\mathbf{h}(\frac{5\pi}{2})$.

(c) Compute $\mathbf{h}'(\pi)$ and $\mathbf{h}'(\frac{5\pi}{2})$.

(d) Draw $\mathbf{h}'(\pi)_{\mathbf{h}(\pi)}$ and $\mathbf{h}'\left(\frac{5\pi}{2}\right)_{\mathbf{h}(\frac{5\pi}{2})}$ on your sketch in part (a).

(5) [**Marked**] Let β be the vector-valued function

$$\beta : (-2,2) \times (0,2\pi) \to \mathbb{R}^3, \qquad \beta(u,v) = \left\{ \frac{3u}{2} - \frac{u\cos(v)}{\sqrt{1+u^2}}, \sin(v), \frac{3u^2}{4} + \frac{\cos(v)}{\sqrt{1+u^2}} \right\}.$$

(a) Sketch the image of β (i.e. plot all values $\beta(u,v)$, for (u,v) in the domain of β).

(b) On the sketch in part (a), indicate (i) the path obtained by holding $v = \pi/2$ and varying u, and (ii) the path obtained by holding $u = 0$ and varying v.

(c) Compute the following quantities:

$$\beta\left(0,\frac{\pi}{2}\right), \qquad \partial_1\beta\left(0,\frac{\pi}{2}\right), \qquad \partial_2\beta\left(0,\frac{\pi}{2}\right).$$

(d) Draw the following tangent vectors on your sketch in part (a):

$$X_1 = \partial_1\beta\left(0,\frac{\pi}{2}\right)_{\beta\left(0,\frac{\pi}{2}\right)}, \qquad X_2 = \partial_2\beta\left(0,\frac{\pi}{2}\right)_{\beta\left(0,\frac{\pi}{2}\right)}.$$

(6) *(Compute 'n' plot)* Let λ denote the vector-valued function

$$\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad \lambda(t) = (t, t^2 - 1).$$

(a) Compute the following: $\lambda(-2)$, $\lambda(-1)$, $\lambda(0)$, $\lambda(1)$, and $\lambda(2)$.

(b) Compute the following: $\lambda'(-2)$, $\lambda'(-1)$, $\lambda'(0)$, $\lambda'(1)$, and $\lambda'(2)$.

(c) Sketch the values $\lambda(t)$, for all $-3 < t < 3$, on a Cartesian plane.

(d) Draw the following tangent vectors as arrows on your sketch in part (a):

$$\lambda'(-2)_{\lambda(-2)}, \qquad \lambda'(-1)_{\lambda(-1)}, \qquad \lambda'(0)_{\lambda(0)}, \qquad \lambda'(1)_{\lambda(1)}, \qquad \lambda'(2)_{\lambda(2)}.$$

(7) *(Compute 'n' plot II)* Consider the vector-valued function

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u).$$

(See also Question 8 from Problem Sheet 1.)

(a) Sketch the image of σ. (Use a computer to help if needed; see the *Additional Resources* section on the *QMPlus* page)

(b) On the sketch in part (a), indicate (i) the path obtained by holding $u = \frac{\pi}{2}$ and varying v, and (ii) the path obtained by holding $v = \frac{\pi}{2}$ and varying u.

(c) Compute the partial derivatives $\partial_1\sigma(u, v)$ and $\partial_2\sigma(u, v)$ for all $(u, v) \in \mathbb{R}^2$.

(d) Draw the following tangent vectors on your sketch in part (a):

$$X_1 = \partial_1\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right)_{\sigma(\frac{\pi}{2}, \frac{\pi}{2})}, \qquad X_2 = \partial_2\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right)_{\sigma(\frac{\pi}{2}, \frac{\pi}{2})}.$$

(8) *(Gradients 'n' plot)* Consider the function

$$p : \mathbb{R}^2 \to \mathbb{R}, \qquad p(x, y) = x - y^2.$$

(a) Sketch the following sets on a Cartesian plane:

(i) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\}$.

(ii) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 2\}$.

(iii) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = -2\}$.

(b) Compute the gradient $\nabla p(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

(c) Plot the following values onto your sketch from part (a):

(i) $\nabla p(0, 0)$.

(ii) $\nabla p(-1, -1)$.

(ii) $\nabla p(-1, 1)$.

(9) *(Connections to "Convergence and Continuity")* Consider the following subsets of $\mathbb{R}^2$:

$$V = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}, \qquad L = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}.$$

(a) Give an informal justification of the following: (i) V is open; (ii) L is not open.

(b) *(Not examinable)* Give a rigorous proof of the two statements in part (a).

(c) Is the following subset of $\mathbb{R}^2$ connected:

$$Q = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}?$$

Give a brief (informal) justification of your answer.

(10) *(Good derivative, bad derivative)*

(a) *(Not examinable)* Give an example of a function $b : \mathbb{R}^2 \to \mathbb{R}$ such that (i) $\partial_1 b(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, but (ii) $\partial_2 b(x, y)$ fails to exist for some (x, y).

(b) *(Fun! But not examinable)* Give an example of a function $b : \mathbb{R}^2 \to \mathbb{R}$ such that (i) $\partial_1 b(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, but (ii) $\partial_2 b(x, y)$ fails to exist for *any* (x, y).