Linear algebra

Throughout work in $\mathbb{R}^n$ (n-dimensional Euclidean space)

If $\underline{x} \in \mathbb{R}^n$ then $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ entries where $x_1, x_2, \ldots, x_n \in \mathbb{R}$

e.g. $\underline{y} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \in \mathbb{R}^3$ where $y_1 = 2$, $y_2 = 4$, $y_3 = 3$

$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ zero vector (dimension clear from context)

If A $n \times m$ matrix (n rows, m columns)

e.g. $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 5 \end{pmatrix}$ Then $A^T = \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 1 & 5 \end{pmatrix}$

Convention

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ is a $n \times 1$ matrix (column vector)

$\underline{x}^T = (x_1 \; x_2 \cdots x_n)$ $1 \times n$ matrix (row vector)

$\underline{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$

$$\underline{x}^T \underline{z} = x_1 z_1 + x_2 z_2 + \cdots x_n z_n$$
$$= \sum x_i z_i$$

(called dot product or scalar product)

For $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in \mathbb{R}^m$ and $a_1, a_2, \dots, a_n \in \mathbb{R}$

$$a_1\underline{x}_1 + a_2\underline{x}_2 + \dots + a_n\underline{x}_n \in \mathbb{R}^m$$

is called <u>linear combination</u> of vectors $\underline{x}_1, \dots, \underline{x}_n$

For $m=1$, $x_1, \dots, x_n \in \mathbb{R}$ are real numbers or variables

linear combination for real numbers or variables is given by

$$a_1x_1 + a_2x_2 + \dots + a_nx_n$$

Can be written concisely as

$$\underline{a}^T\underline{x} \quad \text{where} \quad \underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For constants $a_1, a_2, \dots, a_n \in \mathbb{R}$, $b \in \mathbb{R}$

(1) $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ ← linear equation over variables $x_1, \dots, x_n$

(2) $a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b$ ← linear inequality over variables $x_1, \dots, x_n$

Can write this concisely as (1) $\underline{a}^T\underline{x} = b$

(2) $\underline{a}^T\underline{x} \leq b$

For $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$

Write $\underline{x} \geq \underline{y}$ to mean $x_1 \geq y_1, x_2 \geq y_2, \dots, x_n \geq y_n$

For $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$

Write $\underline{x} \geqslant \underline{y}$ to mean $x_1 \geqslant y_1,\ x_2 \geqslant y_2, \ldots, x_n \geqslant y_n$

True or False $\begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix} \geqslant \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$? True

If $\underline{x}, \underline{y} \in \mathbb{R}^n$ then $\underline{x} \geqslant \underline{y}$ or $\underline{y} \geqslant \underline{x}$ False e.g. $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

If $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$ and $\underline{x} \geqslant \underline{y}$ and $\underline{y} \geqslant \underline{z}$ then $\underline{x} \geqslant \underline{z}$

True.

Suppose A is an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$

$$A\underline{x} = \underline{b}$$

Suppose A is an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$

$A\underline{x} = \underline{b}$ is a concise way of writing

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m \end{aligned}$$

$A\underline{x} \leq \underline{b}$ is a concise way of writing

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &\leq b_m \end{aligned}$$

Example of mathematical program

$$\begin{array}{ll} \text{maximise} & x_1^2 - \sqrt{x_2} + \log(x_1 + x_3) \quad \leftarrow \text{objective function} \\ \text{subject to} & \left.\begin{array}{l} x_1 + x_3^2 \leqslant 2 \\ x_2 + x_1 x_3 \geqslant 5 \\ x_1 + x_3 = 3 \end{array}\right\} \text{constraints} \\ & \underline{x_1 \geqslant 0, \; x_2 \leqslant 0, \; x_3 \text{ unrestricted}} \end{array}$$

goal

↑ variables with sign restrictions

Mark the different elements

<u>Defn</u>

A <u>mathematical program</u> is an optimisation problem consisting of 4 components:

1) A set of variables with possible sign restrictions (always listed at the end)
2) An objective function (i.e. a function of the variables that we want to optimise)
3) A goal (i.e. to maximise or minimise objective function)
4) A set of constraints on the variables given by equations or inequalities.

Defn A <u>linear program</u> is a mathematical program where

- variables assumed to be continuous and real
  (e.g. cannot constrain variable to be integer)
- the objective function is a linear combination of the variables
- each constraint is a linear inequality or linear equation over the variables.

Examples of linear program

Ex1 maximise $2x_1 - 5x_2 + 3x_3$

subject to

$$x_1 + x_2 + x_3 \leq 7$$
$$x_1 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Ex2 minimise $3x_1 - x_2$

subject to

$$-x_1 + 6x_2 + x_3 \geq -3$$
$$x_1 + x_2 = 1$$
$$x_3 \leq 2$$

x_1 unrestricted

$x_2 \geq 0$

$x_3 \leq 0$

Quiz: which of these are linear programmes and which are not?

Which of these is a linear program?

Maximise $2x + 3y - 10z$ ✓

Subject to $x \leq 3$

$$\boxed{4 \leq 2y - z \leq 5 + 3x}$$

$x \geq 0,\ y \leq 0,$

z unrestricted

replace

can write as two separate inequalities, namely

$4 \leq 2y - z$

$2y - z \leq 5 + 3x$

Then rearrange these:

$2y - z \geq 4$

$-3x + 2y - z \leq 7$

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minimise $(2x_1+1)^2 + 2x_2 - 4x_1^2 - 1$ (✓)

subject to

$x_1 + x_2 \geq 3$

$x_1 - x_2 \leq 5$

$x_1, x_2 \geq 0$

After expanding objective function becomes

$4x_1 + 2x_2$

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maximise $2x_1 + 3x_2 - x_3 + 6$ ✗

subject to

$x_1 + 2x_2 \geq 3$

$x_1 + 5x_2 = 10$

x_1, x_2, x_3 unrestricted

objective function not linear combination because of 6.

Defn We say that a linear program is in <u>standard (inequality) form</u> if it is written as

$$\begin{aligned} \text{maximise} \quad & \underline{c}^T \underline{x} \\ \text{subject to} \quad & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

Where A is an $m \times n$ matrix of real numbers
$\underline{c} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$, $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is vector of variables.

Expand above: assume that

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad \underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Above becomes

$$\begin{aligned} \text{maximise} \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \text{subject to} \quad & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \\ & a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \\ & \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_n \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

why do we want to do this?

Write Ex 1 and Ex 2 in standard inequality form

maximise $\underline{c}^T \underline{x}$

subject to $A\underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

Ex1 maximise $2x_1 - 5x_2 + 3x_3$

subject to $x_1 + x_2 + x_3 \leq 7$

$x_1 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

maximise $(2, -5, 3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

subject to $\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\underline{x}} \leq \underbrace{\begin{pmatrix} 7 \\ 5 \end{pmatrix}}_{\underline{b}}$

$\underline{x} \geq \underline{0}$

What about ex 2. Have some problems

maximise $\underline{c}^T \underline{x}$

subject to $A\underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

Ex2 minimise $3x_1 - x_2$

subject to $-x_1 + 6x_2 + x_3 \geq -3$

$x_1 + x_2 = 1$

$x_3 \leq 2$

x_1 unrestricted

$x_2 \geq 0$

$x_3 \leq 0$

How to transform any linear program to standard inequality form

① For each variable x_i if sign constraint is

$x_i \geq 0$ ✓

$x_i \leq 0$ replace x_i with $\bar{x}_i \geq 0$ where $x_i = -\bar{x}_i$

x_i unrestricted replace x_i with $x_i^+ - x_i^-$ with $x_i^+ \geq 0$ $x_i^- \geq 0$

② If goal is $\min \underline{c}^T \underline{x}$ replace with $\max (-\underline{c})^T \underline{x}$

minimizing a fn is same as maximising negative of that function

③ For each constraint, if constraint is

$\underline{a}^T \underline{x} \leq b$ ✓

$\underline{a}^T \underline{x} \geq b$ replace with $(-\underline{a})^T \underline{x} \leq -b$

$\underline{a}^T \underline{x} = b$ replace with two constraints $\underline{a}^T \underline{x} \leq b$, $(-\underline{a})^T \underline{x} \leq -b$

e.g. $r \leq 2$, $-r \leq -2$ $\Leftrightarrow r = 2$

Apply to Ex2

Ex2 minimise $3x_1 - x_2$

subject to
$$-x_1 + 6x_2 + x_3 \geq -3$$
$$x_1 + x_2 = 1$$
$$x_3 \leq 2$$

x_1 unrestricted
$x_2 \geq 0$
$x_3 \leq 0$

①

min $3x_1^+ - 3x_1^- - x_2$

sub to
$$-x_1^+ + x_1^- + 6x_2 - \bar{x}_3 \geq -3$$
$$x_1^+ - x_1^- + x_2 = 1$$
$$-\bar{x}_3 \leq 2$$

$x_1^+, x_1^-, x_2, \bar{x}_3 \geq 0$

$$\boxed{\begin{aligned} x_1 &= x_1^+ - x_1^- \\ x_3 &= -\bar{x}_3 \end{aligned}}$$

① Fixed sign restrictions

$$\begin{aligned} \min \quad & 3x_1^+ - 3x_1^- - x_2 \\ \text{sub to} \quad & -x_1^+ + x_1^- + 6x_2 - \bar{x}_3 \geq -3 \\ & x_1^+ - x_1^- + x_2 = 1 \\ & -\bar{x}_3 \leq 2 \\ & x_1^+, x_1^-, x_2, \bar{x}_3 \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} x_1 &= x_1^+ - x_1^- \\ x_3 &= -\bar{x}_3 \end{aligned}}$$

② Fix goal

$$\begin{aligned} \max \quad & -3x_1^+ + 3x_1^- + x_2 \\ \text{sub to} \quad & -x_1^+ + x_1^- + 6x_2 - \bar{x}_3 \geq -3 \\ & x_1^+ - x_1^- + x_2 = 1 \\ & -\bar{x}_3 \leq 2 \\ & x_1^+, x_1^-, x_2, \bar{x}_3 \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} x_1 &= x_1^+ - x_1^- \\ x_3 &= -\bar{x}_3 \end{aligned}}$$

③ Fix constraints

$$\begin{aligned} \max \quad & -3x_1^+ + 3x_1^- + x_2 \\ \text{sub to} \quad & +x_1^+ - x_1^- - 6x_2 + \bar{x}_3 \leq +3 \\ & x_1^+ - x_1^- + x_2 \leq 1 \\ & -x_1^+ + x_1^- - x_2 \leq -1 \\ & -\bar{x}_3 \leq 2 \\ & x_1^+, x_1^-, x_2, \bar{x}_3 \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} x_1 &= x_1^+ - x_1^- \\ x_3 &= -\bar{x}_3 \end{aligned}}$$

Given optimal solution for transformed program, can use this substitution to get optimal solution to original.

In matrix form $\max \ \underline{c}^T \underline{x}$

sub to $A\underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

$$\underline{x} = \begin{pmatrix} x_1^+ \\ x_1^- \\ x_2 \\ \bar{x}_3 \end{pmatrix} \qquad \underline{c} = \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & -6 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & +1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

Careful about order of variables in vector. If we change order then A and $\underline{c}$ change.

<u>Remark</u> As a convention, when ordering variables,

- put variables in alphabetical order
- lower indices before higher indices
- $+$'s before $-$'s

e.g. $x_1, x_2^+, x_2^-, x_3, x_4, y_1^+, y_1^-, y_2$

The next two pages cover further revision of linear algebra that will be used later in the module.

This was <u>not</u> covered in week 1 but it will help you to do the seminar questions

Matrix multiplication interpreted as linear combinations

When $\underline{x}_1, \dots, \underline{x}_n \in \mathbb{R}^m$ and $a_1, \dots, a_n \in \mathbb{R}$ we have

Ask

$$a_1 \underline{x}_1 + a_2 \underline{x}_2 + \dots + a_n \underline{x}_n = \begin{pmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$= X \underline{a}$$

$m \times n$ matrix whose n columns are the vectors $\underline{x}_1, \dots, \underline{x}_n$

Linear independence and rank.

(will be used in later weeks)

a's are vectors

We say vectors $\underline{a}_1, \dots, \underline{a}_m \in \mathbb{R}^n$ are <u>linearly dependent</u> if $\exists\, x_1, \dots, x_m$ not all zero s.t.

$$x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_m \underline{a}_m = \underline{0}$$

i.e.

$$\begin{pmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \underline{0}$$

call this $n \times m$ matrix A

call this vector $\underline{x} \in \mathbb{R}^m$

i.e. $A\underline{x} = \underline{0}$ for some $\underline{x} \neq \underline{0}$

Show $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ are linearly dependent.

Say $\underline{a}_1, \dots, \underline{a}_m$ are linearly independent if they are not linearly dependent.

So $\underline{a}_1, \dots, \underline{a}_m$ are <u>linearly independent</u> if, whenever $x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_m \underline{a}_m = \underline{0}$,

we have $x_1 = x_2 = \dots = x_m = 0$

i.e. if $A\underline{x} = \underline{0}$ has unique solution $\underline{x} = \underline{0}$

Rank of a matrix $\quad m\overset{n}{\begin{pmatrix} \quad \end{pmatrix}}$

For $m \times n$ matrix A, its n columns are vectors in $\mathbb{R}^m$
its m rows are vectors in $\mathbb{R}^n$

column rank$(A) = \max\{k :$ A has k linearly independent columns$\}$

row rank$(A) = \max\{l :$ A has l linearly independent rows$\}$

<u>Fact</u> row rank (A) = column rank(A)

Just call it rank(A).

$\text{rank}(A) \leq \min(m, n)$

<u>Fact</u> If A is $n \times n$ matrix then

rank$(A) = n \iff A$ is invertible

$\iff A\underline{x} = \underline{b}$ has unique solution for every $\underline{b} \in \mathbb{R}^n$.