Actuarial Mathematics II
MTH5125

Revision: Survival Models and Life Tables
Chapter 2 and 3 (DHW)

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The future lifetime random variable

- **Status** (x):

$$(x) \stackrel{\text{not.}}{=} \text{a life aged } x, \qquad x \geq 0$$

- **Future lifetime** of (x):

$$T_x$$

  - Assumption: T_x is a *continuous* r.v. on $(0, +\infty)$.

- **Age-at-death** of (x):

$$x + T_x$$

- **Lifetime distribution** of (x):

$$F_x(t) = \mathbb{P}[T_x \leq t]$$

- **Survival function** of (x):

$$S_x(t) = 1 - F_x(t)$$

The future lifetime random variable

Consider a person (x) with:

- Current future lifetime: T_x
- Future lifetime at birth: T_0
- Future lifetime at age $y \geq x$

For any $y > x$ and any $t > 0$:

$$P\left[T_y \leq t\right] = P\left[T_0 \leq y + t \middle| T_0 > y\right]$$

For any $t, u > 0$:

$$P\left[T_{x+t} \leq u\right] = P\left[T_x \leq t + u \middle| T_x > t\right]$$

The future lifetime random variable

- Lifetime distributions F_x and F_0:

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)} \tag{2.2}$$

- Survival functions S_x and S_0:

$$S_0(x+t) = S_0(x)\ S_x(t) \tag{2.4}$$

- Survival functions S_{x+t} and S_x:

$$S_x(t+u) = S_x(t)\ S_{x+t}(u) \tag{2.5}$$

The future lifetime random variable

(2.2) comes from:

$$\begin{aligned} F_x(t) &= P[T_x \le t] \\ &= P[T_0 \le x+t \mid T_0 > x] \\ &= \frac{P[(T_0 \le x+t) \cap (T_0 > x)]}{P[T_0 > x]} \\ &= \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} \\ &= \frac{F_0(x+t) - F_0(x)}{S_0(x)} \end{aligned}$$

The future lifetime random variable

- Consider (x) with continuous future lifetime T_x.
- $S_x(t)$ is a **survival function** for (x) if and only if the following conditions are satisfied:
  - Condition 1:
    $$S_x(0) = 1$$
  - Condition 2:
    $$\lim_{t \to +\infty} S_x(t) = 0$$
  - Condition 3:
    $S_x(t)$ is a non-increasing continuous function of t

The future lifetime random variable

- For all **survival functions** $S_x(t)$ in this course, we make the following assumptions:
  - Assumption 1:
$$\frac{d}{dt} S_x(t) \text{ exists for all } t > 0$$
  - Assumption 2:
$$\lim_{t \to +\infty} t \, S_x(t) = 0$$
  - Assumption 3:
$$\lim_{t \to +\infty} t^2 \, S_x(t) = 0$$
- Assumptions 2 and 3 ensure that the mean and the variance of the distribution of T_x exist.

The force of mortality

Consider a person with survival function at birth $P\left[T_0 > t\right]$.

The force of mortality at age x :

$$\mu_x \stackrel{\text{def}}{=} \lim_{dx \to 0^+} \frac{P\left[T_x < dx\right]}{dx}$$

$$\Leftrightarrow$$

$$\mu_x = \lim_{dx \to 0^+} \frac{P\left[T_0 \leq x + dx \middle| T_0 > x\right]}{dx}$$

$$\Leftrightarrow$$

$$\mu_x = \lim_{dx \to 0^+} \frac{1 - S_x\left(dx\right)}{dx}$$

The force of mortality

If

$$\mu_x = \lim_{dx \to 0^+} \frac{1 - S_x(dx)}{dx}$$

and

$$S_0(x + dx) = S_0(x)\, S_x(dx)$$

or

$$S_x(dx) = \frac{S_0(x + dx)}{S_0(x)}$$

then

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{d_x} \frac{S_0(x) - S_0(x + dx)}{S_0(x)}$$

The force of mortality

$$\begin{aligned}
\mu_x &= \lim_{dx \to 0^+} \frac{1}{d_x} \frac{S_0(x) - S_0(x+dx)}{S_0(x)} \\
&= \frac{1}{S_0(x)} \lim_{dx \to 0^+} \frac{S_0(x) - S_0(x+dx)}{dx} \\
&= \frac{1}{S_0(x)} \left(-\frac{d}{dx} S_0(x) \right)
\end{aligned}$$

The force of mortality

- μ_x in terms of S_0:

$$\boxed{\mu_x = -\frac{1}{S_0(x)}\frac{d}{dx}S_0(x)} \tag{2.9}$$

- The pdf of T_x:

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t)$$

- μ_x in terms of f_0 and S_0:

$$\boxed{\mu_x = \frac{f_0(x)}{S_0(x)}}$$

The force of mortality

Suppose that x is fixed and t is variable, then:

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \tag{2.10}$$

or

$$\mu_{x+t}\, dt \approx P\left[T_x \leq t + dt \middle| T_x > t\right]$$

An expression for $S_x(t)$:

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s} ds\right) \tag{2.11}$$

The force of mortality

(2.11) comes from:

$$\begin{aligned}\mu_x &= \frac{-1}{S_0(x)}\frac{d}{dx}S_0(x)\\ &= -\frac{d}{dx}\ln S_0(x)\end{aligned}$$

$$\Leftrightarrow$$

$$\int_0^y \mu_x dx = -\left[\ln S_0(y) - \ln S_0(0)\right] = -\ln S_0(y)$$

We used that $\ln S_0(0) = \ln\left(P\left[T_0 > 0\right]\right) = \ln(1) = 0$

The force of mortality

Hence:

$$S_0\left(y\right) = \exp\left(-\int_0^y \mu_x dx\right)$$

and

$$S_x\left(t\right) = \exp\left(-\int\limits_0^t \mu_{x+s} ds\right)$$

Mortality Laws

- **Gompertz' law of mortality**:

$$\boxed{\mu_x = Bc^x, \qquad x > 0}$$

where B and c are constants such that $B > 0$ and $c > 1$.

- **Makeham's law of mortality**:

$$\boxed{\mu_x = A + Bc^x, \qquad x > 0}$$

where A, B and c are constants such that A, $B > 0$ and $c > 1$.

- Both models often provide a good fit to mortality data over certain age ranges, particularly from middle age to early old age.

Actuarial Notations

- **Survival rates**:

$$ {}_t p_x \overset{\text{not.}}{=} \mathbb{P}[T_x > t] = S_x(t) \qquad (2.13) $$

- **Mortality rates**:

$$ {}_t q_x \overset{\text{not.}}{=} \mathbb{P}[T_x \leq t] = F_x(t) \qquad (2.14) $$

- Deferred mortality rates:

$$ {}_{u|t} q_x \overset{\text{not.}}{=} \mathbb{P}[u < T_x \leq u + t] = S_x(u) - S_x(u + t) \qquad (2.15) $$

- Simplified notations for 1 - year probabilities:

$$ p_x \overset{\text{not.}}{=} {}_1 p_x $$

$$ q_x \overset{\text{not.}}{=} {}_1 q_x $$

$$ {}_{u|} q_x \overset{\text{not.}}{=} {}_{u|1} q_x $$

Actuarial notations

- Survival and mortality rate add to 1:

$$ {}_tp_x + {}_tq_x = 1 $$

- Survival rates at different ages:

$$ {}_{t+u}p_x = {}_tp_x \times \ {}_up_{x+t} \tag{2.16} $$

- Survival rates in terms of one-year survival rates:

$$ {}_np_x = p_x \times p_{x+1} \times \ldots \times \ p_{x+n-1} $$

- Deferred mortality rates:

$$ {}_{u|t}q_x = {}_up_x - {}_{u+t}p_x = {}_up_x \times {}_tq_{x+u} $$

Actuarial notations

- Force-of-mortality at age x:

$$\mu_x = \lim_{h \to 0+} \frac{{}_h q_x}{h} = -\frac{1}{{}_x p_0} \frac{d}{dx} {}_x p_0 \tag{2.17}$$

- Force-of-mortality at age $x + t$:

$$\mu_{x+t} = -\frac{1}{{}_t p_x} \frac{d}{dt} {}_t p_x \tag{2.18}$$

- Density function of T_x:

$$f_x(t) = {}_t p_x \, \mu_{x+t} \tag{2.19}$$

- Survival rate in terms of forces-of-mortality:

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds \right) \tag{2.20}$$

Actuarial notations

- Death rates and forces-of-mortality:

$$ {}_tq_x = \int_0^t {}_sp_x \; \mu_{x+s} ds \qquad (2.21) $$

- Graphical interpretation:

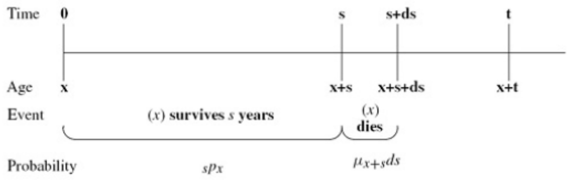

Figure 2.3 Time-line diagram for $_tq_x$

- Approximation:

$$ q_x \approx \mu_{x+\frac{1}{2}}, \qquad \text{when } q_x \text{ is small} $$

Actuarial notations

Complete expectation of life:

$$\mathring{e}_x = E\left[T_x\right] = \int_0^t \ {}_t p_x dt$$

Curtate future lifetime:

$$K_x \stackrel{\text{def}}{=} \lfloor T_x \rfloor$$

Probability function of K_x :

$$P\left[K_x = k\right] = \ {}_k p_x \ q_{x+k} \text{ for } k = 0, 1, 2, 3, \ldots$$

Curtate expectation of life:

$$e_x = E\left[K_x\right] = \sum_{k=1}^{\infty} \ {}_k p_x$$

- A **survival model** is set of forces-of-mortality

$$\{\mu_y \mid y \geq x_0\}$$

used to determine the survival probabilities of a specified group of persons.

- For any person (x), $x \geq x_0$, belonging to this specified group, μ_y, $y \geq x$, is his assumed force-of-mortality at age y.
- Example of a survival model: **Makeham survival model**:

$$\mu_y = A + Bc^y, \qquad \text{for all } y \geq x_0$$

- Examples of a specified group:
  - All male smokers of age $x \geq x_0$ who underwrite this year a life insurance contract with insurer A.
  - All female persons of age $x = x_0$ who underwrite this year a lifelong pension contract with insurer B.

Standard Ultimate Survival Model (used in DHW)

- Makeham Survival Model with

$$\begin{aligned} A &= 0.00022 \\ B &= 2.7 \times 10^{-6} \\ C &= 1.124 \end{aligned}$$

- The force of mortality increases exponentially with age as $c > 1$
- $B > 0$: ensures that force of mortality is positive
- $A > 0$: reflects the risk of accidental death

Life tables

- Consider the following *survival model*:

$$\boxed{\{\mu_y \mid y \geq x_0\}}$$

- Survival probabilities for (x) whose survival probabilities follow from this model:

$${}_tp_{x+u} = \exp\left(-\int_0^t \mu_{x+u+s}\, ds\right), \qquad x \geq x_0 \text{ and } u \geq 0$$

- **Life table** constructed from this survival model:
    - l_{x_0} = arbitrary positive number, called the **radix**.
    - For $t \geq 0$, define l_{x_0+t} by

$$\boxed{l_{x_0+t} = l_{x_0} \times {}_tp_{x_0}}$$

- Interpretation of l_{x+t}:
  - Let $\mathbf{L}_{x+t}$ be the number of survivors at age $x+t$ from a closed group of l_x persons of age x, with survival probabilities following from the survival model $\{\mu_y \mid y \geq x_0\}$.
  - The expected number of survivors:

$$\boxed{\mathbb{E}[\mathbf{L}_{x+t}] = l_{x+t}}$$

- Interpretation of d_x:
  - Let $\mathbf{D}_x$ be the number of deaths in the year of age x to $x+1$ from the same closed group of l_x persons of age x.
  - The expected number of deaths:

$$\boxed{\mathbb{E}[\mathbf{D}_x] = d_x}$$

Constructing the Standard Ultimate Survival Table (DHW)
Assume that μ_x follows Makeham's Law:

$$\mu_x = A + Bc^x$$

Derive an expression for $S_x(t) \equiv {}_t p_x$

$$\begin{aligned} S_x(t) &= \exp\left\{ -\int_x^{x+t} (A + Bc^r)\, dr \right\} \\ &= \exp\left(-At - B \int_x^{x+t} \exp\{r \ln c\}\, dr \right) \\ &= \exp\left(-At - \frac{B}{\ln c}\left(c^{x+t} - c^x\right) \right) \\ &= \exp\left(-At - \frac{B}{\ln c} c^x (c-1) \right) \end{aligned}$$

Deriving the Life table

Derive an expression for $S_x(1) \equiv p_x$

$$ {}_1p_x = p_x = \exp\left(-A - \frac{B}{\ln c} c^x (c-1)\right) $$

Life tables

- Consider (x), $x \geq x_0$, who follows the survival model $\{\mu_y \mid y \geq x_0\}$, with corresponding life table $\{l_y \mid y \geq x_0\}$.
- Survival probabilities for (x):

$$\boxed{{}_t p_x = \frac{l_{x+t}}{l_x}} \quad \text{for any } t \geq 0 \tag{3.1}$$

- Notation:

$$d_x \overset{\text{not.}}{=} l_x - l_{x+1} \tag{3.4}$$

- One-year mortality rates:

$$\boxed{q_x = \frac{d_x}{l_x}} \tag{3.5}$$

- Deferred mortality rates:

$$\boxed{{}_{t|u} q_x = \frac{l_{x+t} - l_{x+t+u}}{l_x}}$$

Deriving the Life table

Derive an expression for $S_x(1) \equiv p_x$

$${}_1p_x = p_x = \exp\left(-A - \frac{B}{\ln c} c^x (c-1)\right)$$

Derive an expression for l_x

$$l_{x+1} = l_x p_x$$

Derive an expression for d_x

$$d_x = l_x - l_{x+1}$$

Derive an expression for q_x

$$q_x = \frac{d_x}{l_x}$$

Life tables

- Relations:

$$\boxed{{}_tp_x = \frac{l_{x+t}}{l_x}}, \quad \boxed{q_x = \frac{d_x}{l_x}} \quad \text{and} \quad \boxed{{}_{t|u}q_x = \frac{l_{x+t} - l_{x+t+u}}{l_x}}$$

- Stochastic interpretation:
  - p - and q - functions are *probabilities*.
  - l - and d - functions are *expected numbers* of survivors and dyers from a closed group of l_x persons.
- Deterministic interpretation:
  - p - and q - functions are *fractions*.
  - l - and d - functions are *observed numbers* of survivors and dyers from a closed group of l_x persons.