Mathematics of Assest Management
MTH6113

Topic 1

Utility Theory

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Plan

- Consumer's Decision Problem and Utility Theory
  - consumer's preferences and indifference curves
  - utility function
  - decision problem and optimisation

Consumer's Decision Problem

The consumer's decision problem is characterised by:

- **the consumer's preferences**
- **the budget constraint** which defines different consumption bundles that consumer can afford
  - **a bundle**: a particular combination of two or more goods
- **the optimisation problem**
  - how consumer decides which consumption bundle to choose, given her preferences and budget constraint

Consumer's Decision Problem

Consumer's Preferences

An agent has preferences over a choice set X

For example: the choice set is: {apples, bananas}

- M *prefers* {2 apples and 3 bananas} to {1 apple and 1 banana}
- M is *indifferent* between {$\frac{1}{2}$apple and 2 bananas} and {1 apple and 1 banana}

Consumer's Decision Problem

1. Preference relation: a binary relation, $\succsim$, on the choice set X that allows the decision maker to comapre different alternative $\mathbf{x}, \mathbf{y} \in X$ (which can be $\subset \mathbb{R}^n_+$).

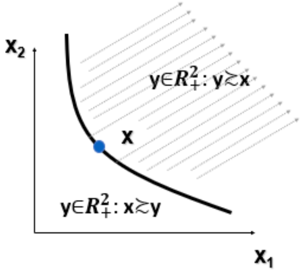
- If $\mathbf{x} \succsim \mathbf{y}$ we say that "$\mathbf{x}$ is at least as good as $\mathbf{y}$" for this decision maker.

Further we can define two other important relations on X :

2. The strict preference relation, $\succ$, defined as: $\mathbf{x} \succ \mathbf{y} \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$ and not $\mathbf{y} \succsim \mathbf{x}$; if $\mathbf{x} \succ \mathbf{y}$ we say that "$\mathbf{x}$ is preferred to $\mathbf{y}$" by the decision maker.
3. The indifference relation, $\sim$, defined as:$\mathbf{x} \sim \mathbf{y} \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{x}$; if $\mathbf{x} \sim \mathbf{y}$ we say that "$\mathbf{x}$ is indifferent to $\mathbf{y}$"

Consumer's Decision Problem

Indifference curve: a set of bundles among which the consumer is indifferent



Assumptions on consumers' preferences:

1. *Completeness*: either $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{y} \succ \mathbf{x}$ or $\mathbf{x} \sim \mathbf{y}$
   - I can always rank goods
2. *Transitivity*: $\mathbf{x} \succ \mathbf{y} \succ \mathbf{z}$ then $\mathbf{z} \not\succ \mathbf{x}$
   - There are no logical inconsistencies

We say that a decision maker with preferences satisfying completeness and transitivity is **rational**

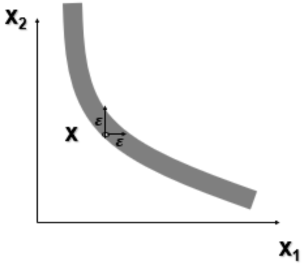
Assumptions on consumers' preferences:

3. *Monotonicity* for all $\mathbf{x}, \mathbf{y} \in X \subset \mathbb{R}^n_+$, if $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$ implies $\mathbf{y} \succ \mathbf{x}$.

3'. *Local Non-satiation (more is better)*

- If for all $\mathbf{x} \in X$ and every $\varepsilon > 0$, there is $\mathbf{y} \in X$ such that $||\mathbf{y} - \mathbf{x}|| \leq \varepsilon$ and $\mathbf{y} \succ \mathbf{x}$.
  - note that $||\ ||$ represents the Euclidian distance between two points.

Implication of local non-satiation: Indifference Curves cannot be thick



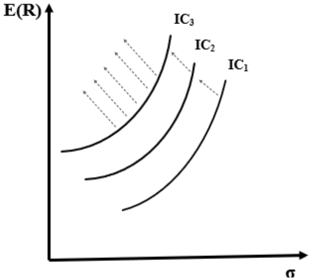
Consumer's Decision Problem

Local non-satiation and investment decisions

- **x** is a good - something that we want to always consume more
- **y** is a "bad": e.g. polution

Investment theory - decision maker wishes to select a portfolio with high expected return and low risk (standard deviation)

- Indifference curve:



Local-nonsatiation will make the decision maker select a portfolio moving N-W

Consumer's Decision Problem

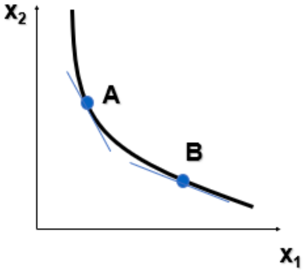
Marginal rate of substitution (MRS): rate at which consumer is willing to substitute one unit of one good for the other good keeping the same level of satisfaction

- the absolute value of the slope of the indifference curve

Consumer's Decision Problem

Diminishing marginal rate of substitution

- the more of good x you have, the more you are willing to give it up to get a little of good y



Utility Function

A rational preference relation can be represented by a **utility function**

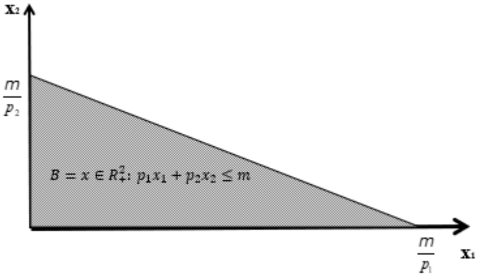
- Utility function: numerical representation: $u : X \to \mathbb{R}$
  - Move from real objects/goods/things to numbers
  - It measures the level of satisfaction that a consumer receives from any bundle
- We can use Maths to find our optimum level of consumption!

Consumer's Decision Problem

Budget set of the consumer - the set of affordable bundles/commodities

$$B = \left[\mathbf{x} \in \mathbf{X} : \mathbf{p}'\mathbf{x} \leq m\right]$$

In $\mathbb{R}^2_+$ the budget set B is depicted in fig.



Consumer's Decision Problem

The optimisation problem of a consumer can be written now as:

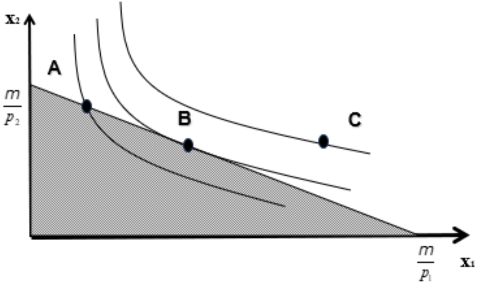
$$\max_{\mathbf{x}} u\left(\mathbf{x}\right)$$

such that the chosen commodities are affordable (in the budget set) or the budget constraint is satisfied:

$$\mathbf{p}'\mathbf{x} \leq m$$

Consumer's Decision Problem

In R^2_+ this problem can be seen diagramatically as:



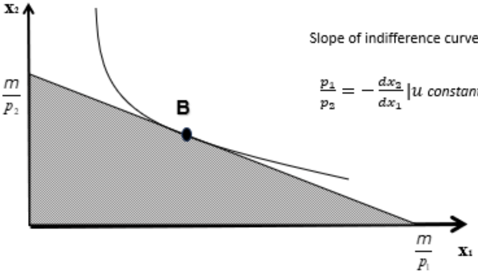
Consumer's Decision Problem

In R^2_+ this problem can be seen diagramatically as:

Slope of budget constraint $= -\frac{p_1}{p_2}$

Slope of indifference curve $= \frac{dx_2}{dx_1}|u\ constant$

$\frac{p_1}{p_2} = -\frac{dx_2}{dx_1}|u\ constant \equiv MRS_{ij}$



How we solve this problem with Calculus
Constrained optimisation use the Lagrangian Method:
Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda\left(m - \mathbf{p}'\mathbf{x}\right)$$

where λ is the Langrange multiplier.
Differentiating the Lagrangian with respect to $\mathbf{x}$ gives us the first order conditions:

$$\frac{\partial u(\mathbf{x})}{\partial x_i} - \lambda p_i = 0 \text{ for all } i = 1, ..., n.$$

Consumer's Decision Problem

If we divide the ith first order condition to the jth order condition:
At the optimum:

$$\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_i}}{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}} = \frac{p_i}{p_j} \text{ for all } i, j = 1, ..., n.$$

- these are necessary conditions for a local optimum.
- they are also sufficient conditions if $u(.)$ is monotone and quasiconcave.
- second order condition can be written as $\mathbf{y}'H(\mathbf{x})\,\mathbf{y} \leq \mathbf{0}$ for all y such that $\mathbf{p}'\mathbf{y} = \mathbf{0}$.
    - Hessian matrix of the utility function is negative semidefinite for all vectors **y** orthogonal to the price vector.

Consumer's Decision Problem - Cobb-Douglas utility

The consumer's optimisation problem is:

$$\max_{x_1, x_2} u\left(x_1, x_2\right) = x_1^a x_2^b \text{ subject to}$$

$$p_1 x_1 + p_2 x_2 \leq m$$

The Langrangian function in this case is:

$$\mathcal{L}\left(\lambda, x_1, x_2\right) = x_1^a x_2^b + \lambda\left(m - p_1 x_1 - p_2 x_2\right)$$

Consumer's Decision Problem - Cobb-Douglas utility

First order conditions:

$$\begin{aligned} a x_1^{a-1} x_2^{b} - \lambda p_1 &= 0 \\ b x_1^{a} x_2^{b-1} - \lambda p_2 &= 0 \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

This system can be simplified to:

$$\begin{aligned} \frac{a x_2}{b x_1} &= \frac{p_1}{p_2} \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

Solution:

$$
\begin{aligned}
x_1^*\left(p_1, p_2, m\right) &= \frac{m}{p_1}\frac{a}{a+b} \\
x_2^*\left(p_1, p_2, m\right) &= \frac{m}{p_2}\frac{b}{a+b}
\end{aligned}
$$

Note that when $a+b=1$ the market demands are equal to the share of income that the consumer allocates to each good.

Consumer's Decision Problem - Cobb-Douglas utility

The second order condition for a local maximum can be written in terms of Bordered Hessian:

$$\begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{pmatrix}$$

Remember from optimisation that the sufficient condition for a local maximum are that the leading principal minors alternate in signe starting with the third being positive.
As $u_{11}, u_{22} < 0$ and $u_{12} = u_{21} > 0$ we need :

$$\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{vmatrix} > 0$$

Note that the determinant above is equal to
$p_1 p_2 u_{21} + p_1 p_2 u_{12} - p_2^2 u_{11} - p_1^2 u_{22} > 0$