Actuarial Mathematics II
MTH5125

Revision: Insurance benefits and Annuities
Chapter 4 and 5 (DHW)

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Spring Term

- Amicable Society for a Perpetual Assurance Office:
  - Founded in London, 1706.
  - First company offering life insurance.
- Society for Equittable Assurances on Lives and Survivorship:
  - Also known as Equittable Life, founded in London, 1762.
  - World's oldest mutual life insurer.

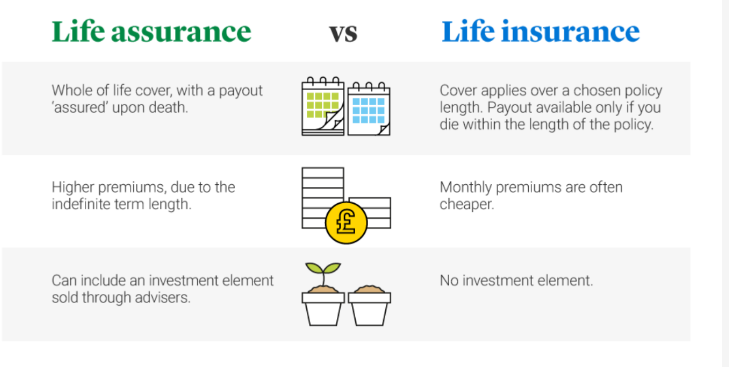
Introduction to life insurance

Samuel Huebner (1882-1964)

- One of the first insurance economists.
- The Economics of Life Insurance(1927):
  - *Not to insure adequately through life insurance is to gamble with the greatest economic risk confronting man. If understood, the gamble is a particularly selfish one, since the blow, in the event the gamble is lost, falls upon an innocent household whose economic welfare should have been the family head's first consideration.*

Life insurance vs life assurance

- https://www.legalandgeneral.com/insurance/life-insurance/definitions/assurance-vs-insurance/



- In DHW - insurance/assurance

Insurance benefits - assumptions

- Technical basis = a set of assumptions used for performing life insurance or pension calculations.
- Technical basis in this chapter (used in the examples):
  - The Standard Ultimate Survival Model:

$$\mu_x = 0.00022 + 2.7 \times 10^{-6} \times 1.124^x$$

  - A constant interest.
  - These are (pedagogically) convenient assumptions.
- Conventions:
  - Time 0 = now.
  - Time unit is 1 year.

Insurance benefits - assumptions

Some notions of financial algebra

- i = annual rate of interest.
- $i^{(p)}$ = nominal interest (compounded p times per year):

$$\left(1+\frac{i^{(p)}}{p}\right)^p = 1+i$$

- δ = force of interest:

$$\delta = \ln(1+i)$$

- v = yearly discount factor:

$$v = \frac{1}{1+i} = e^{-\delta}$$

- d = discount rate per year:

$$d = 1 - v = i\ v = 1 - e^{-\delta}$$

- $d^{(p)}$ = nominal discount rate (compounded p times per year):

$$d^{(p)} = p\left(1 - v^{\frac{1}{p}}\right) = i^{(p)}\ v^{1/p}$$

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Whole life insurance - the continuous case (db payable at instant of death)

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at T_x.
- Benefit cash flow:

$$(1, T_x)$$

- Present value:

$$Z = v^{T_x} = e^{-\delta T_x}$$

- Actuarial value (or EPV):

$$\boxed{\overline{A}_x \overset{\text{not.}}{=} \mathbb{E}\left[e^{-\delta T_x}\right] = \int_0^\infty e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt} \qquad (4.1)$$

Whole life insurance - the continuous case

$$\boxed{\overline{A}_x = \int_0^\infty e^{-\delta s} {}_s p_x \ \mu_{x+s} \ ds}$$

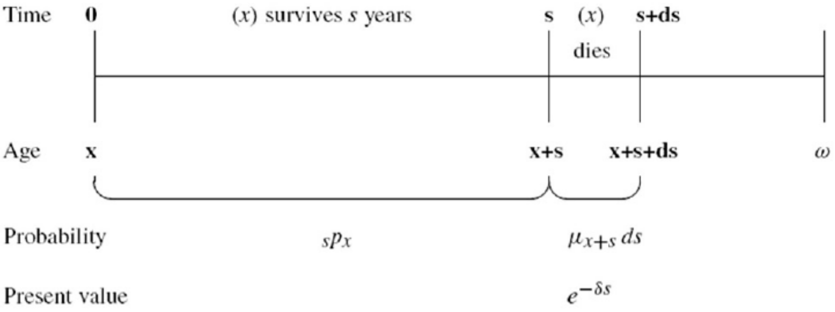

Figure 4.1 Time-line diagram for continuous whole life insurance.

Whole life insurance - the annual case (db payable at end of year of death)

- Consider a life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x + 1$.
- Benefit cash flow:

$$(1, K_x + 1)$$

- Present value:

$$Z = v^{K_x+1}$$

- Actuarial value:

$$\boxed{A_x \overset{\text{not.}}{=} \mathbb{E}[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} \;_{k|}q_x} \tag{4.4}$$

Whole life insurance - the annual case

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \ {}_{k|}q_x$$

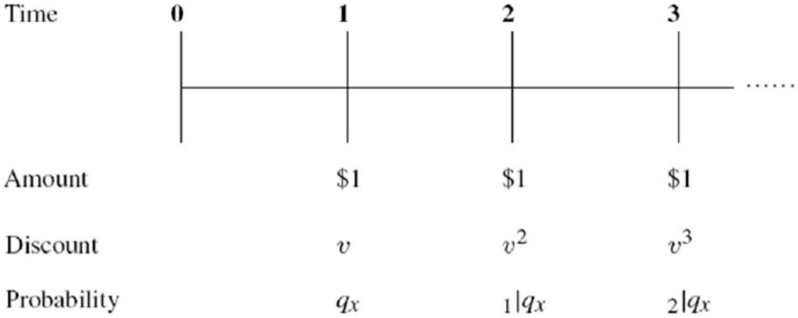

Figure 4.2 Time-line diagram for discrete whole life insurance.

Continuous case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at T_x, provided $T_x \leq n$.
- Benefit cash flow:

$$\left(1_{\{T_x \leq n\}}, T_x\right)$$

- Present value:

$$Z = e^{-\delta T_x} \; 1_{\{T_x \leq n\}}$$

- Actuarial value:

$$\boxed{\bar{A}^{1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \int_0^n e^{-\delta t} \, {}_t p_x \; \mu_{x+t} \; dt} \qquad (4.9)$$

Term insurance (policies of duration n): annual case

Annual case

- Consider a term life insurance underwritten on (x) at time 0, with a payment of 1 at $K_x + 1$, provided $K_x + 1 \leq n$.
- Benefit cash flow:

$$\left(1_{\{K_x+1\leq n\}},\ K_x + 1\right)$$

- Present value:

$$Z = v^{K_x+1}\ 1_{\{K_x+1\leq n\}}$$

- Actuarial value:

$$\boxed{A^{1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x} \qquad (4.10)$$

Pure endowment

- Consider a pure endowment insurance underwritten on (x) at time 0, with a payment of 1 at time n, provided $T_x > n$.
- Benefit cash flow:

$$\left(1_{\{T_x > n\}},\ n\right)$$

- Present value:

$$Z = v^n\, 1_{\{T_x > n\}}$$

- Actuarial value:

$$\boxed{{}_nE_x \overset{\text{not.}}{=} \mathbb{E}[Z] = v^n\, {}_np_x} \tag{4.13}$$

- Alternate notation:

$$A_{x:\overset{\,1}{\overline{n}|}}$$

Continuous case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time T_x, provided $T_x \leq n$, and a payment of 1 at time n, provided $T_x > n$.
- Benefit cash flow:

$$(1, \min(T_x, n)) = \left(1_{\{T_x \leq n\}}, T_x\right) + \left(1_{\{T_x > n\}}, n\right)$$

- Present value:

$$Z = v^{\min(T_x, n)}$$

- Actuarial value:

$$\boxed{\overline{A}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \overline{A}^{\,1}_{x:\overline{n}|} + {}_nE_x} \tag{4.17}$$

Endowment insurance: annual case

- Consider an endowment insurance underwritten on (x) at time 0, with a payment of 1 at time $K_x + 1$, provided $K_x + 1 \leq n$, and a payment of 1 at time n, provided $K_x + 1 > n$.
- Benefit cash flow:

$$(1, \min(K_x + 1, n)) = \left(1_{\{K_x+1 \leq n\}},\ K_x + 1\right) + \left(1_{\{K_x+1>n\}},\ n\right)$$

- Present value:

$$Z = v^{\min(K_x+1, n)}$$

- Actuarial value:

$$\boxed{A_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = A^{1}_{x:\overline{n}|} + {}_nE_x} \qquad (4.19)$$

- Consider a deferred term insurance underwritten on (x) at time 0, with a payment of 1 at T_x, provided $u < T_x \leq u + n$.
- Benefit cash flow:

$$\left(1_{\{u<T_x\leq u+n\}}, T_x\right)$$

- Present value:

$$Z = e^{-\delta T_x} \; 1_{\{u<T_x\leq u+n\}}$$

- Actuarial value:

$$\boxed{{}_{u|}\overline{A}^{\,1}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Z] = \int_u^{u+n} e^{-\delta t}\, {}_tp_x \; \mu_{x+t} \; dt} \qquad (4.21)$$

Annuities

- **Life annuity**: series of payments as long as a given person is alive on the payment dates.
- Payments:
  - at regular intervals,
  - (usually) of the same amount.
- Used for calculating:
  - pension benefits,
  - premiums,
  - policy values.

Annuities-certain

- Annuity-due:

$$\ddot{a}_{\overline{n}|} = 1 + v + ... + v^{n-1} = \frac{1 - v^n}{d} \tag{5.1}$$

- Annuity-immediate:

$$a_{\overline{n}|} = v + ... + v^n = \frac{1 - v^n}{i}$$

- Continuous annuity:

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{1 - v^n}{\delta} \tag{5.2}$$

- Annuity-due with $1/m$-thly payments:

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}\left(1 + v^{\frac{1}{m}} + ... + v^{n-\frac{1}{m}}\right) = \frac{1 - v^n}{d^{(m)}}$$

- Annuity-immediate with $1/m$-thly payments:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\left(v^{\frac{1}{m}} + ... + v^{n-\frac{1}{m}} + v^n\right) = \frac{1 - v^n}{i^{(m)}}$$

Annual life annuities: whole-life annuity-due

- Consider an annuity underwritten to (x) at time 0. It pays 1 annually in advance as long as (x) is alive.
- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

$$Y = 1 + v + ... + v^{K_x} = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - v^{K_x+1}}{d}$$

- Actuarial value:

$$\boxed{\ddot{a}_x \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \frac{1-\mathbb{E}\left[v^{K_x+1}\right]}{d} = \frac{1-A_x}{d}} \qquad (5.3)$$

Annual life annuities: whole-life annuity-due

- Benefit cash flow:

$$\sum_{t=0}^{K_x} (1,\ t) = \sum_{t=0}^{\infty} \left(1_{\{T_x>t\}}, t\right)$$

- Present value:

$$Y = \sum_{t=0}^{\infty} v^t \ 1_{\{T_x>t\}}$$

- Actuarial value:

$$\boxed{\ddot{a}_x = \mathbb{E}\left[Y\right] = \sum_{t=0}^{\infty} v^t \ {}_tp_x} \tag{5.5}$$

Annual life annuities: whole-life annuity-due

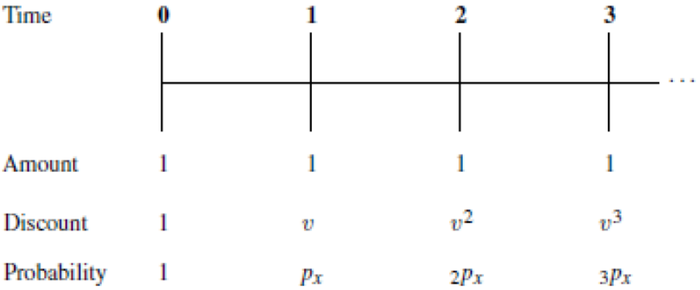

Figure 5.1 Time-line diagram for whole life annuity-due.

Annual life annuities: whole-life annuity-due

- Benefit cash flow:

$$\sum_{k=0}^{K_x} (1, k)$$

- Present value:

$$Y = \ddot{a}_{\overline{K_x+1|}}$$

- Actuarial value:

$$\boxed{\ddot{a}_x = \mathbb{E}\left[Y\right] = \textstyle\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \times {}_{k|}q_x} \qquad (5.6)$$

Annual life annuities: term annuity-due

- Consider an annuity underwritten to (x) at time 0. It pays 1 at times $0, 1, \ldots, n-1$, provided (x) is alive.
- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x, n-1)} (1, t)$$

- Present value:

$$Y = 1 + v + \ldots + v^{\min(K_x, n-1)} = \ddot{a}_{\overline{\min(K_x+1, n)}|} = \frac{1 - v^{\min(K_x+1, n)}}{d}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}[Y] = \frac{1 - \mathbb{E}\left[v^{\min(K_x+1, n)}\right]}{d} = \frac{1 - A_{x:\overline{n}|}}{d}} \qquad (5.7)$$

Annual life annuities: term annuity-due

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x,n-1)} (1,t) = \sum_{t=0}^{n-1} \left(1_{\{T_x>t\}}, t\right)$$

- Present value:

$$Y = \sum_{t=0}^{n-1} v^t \; 1_{\{T_x>t\}}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y\right] = \textstyle\sum_{t=0}^{n-1} v^t \;\; {}_t p_x} \tag{5.8}$$

Annual life annuities: term annuity-due

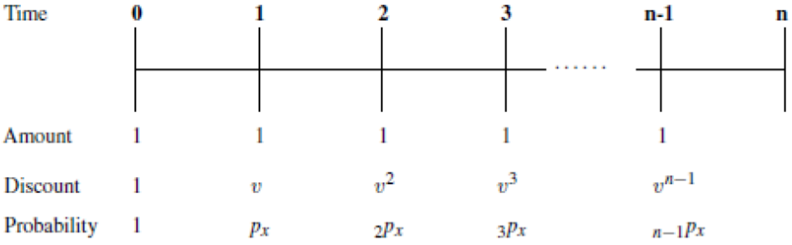

Figure 5.2 Time-line diagram for term life annuity-due.

Annual life annuities: term annuity-due

- Benefit cash flow:

$$\sum_{t=0}^{\min(K_x,n-1)} (1,t)$$

- Present value:

$$Y = \ddot{a}_{\overline{min(K_x+1,n)|}}$$

- Actuarial value:

$$\boxed{\ddot{a}_{x:\overline{n}|} = \mathbb{E}\left[Y\right] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \times {}_{k|}q_x + {}_{n}p_x \times \ddot{a}_{\overline{n}|}}$$

Annual life annuities: immediate life annuities

- Consider a **whole life immediate annuity** underwritten to (x) at time 0. It pays 1 annually in arrear, as long as (x) is alive.
- Benefit cash flow:

$$\sum_{t=1}^{K_x} (1, t) = \sum_{t=1}^{\infty} \left(1_{\{T_x > t\}}, t\right)$$

- Present value:

$$Y^* = \sum_{t=1}^{\infty} v^t \; 1_{\{T_x > t\}}$$

- Actuarial value:

$$\boxed{a_x \overset{\text{not.}}{=} \mathbb{E}\left[Y^*\right] = \ddot{a}_x - 1} \qquad (5.9)$$

Annual life annuities: immediate life annuities

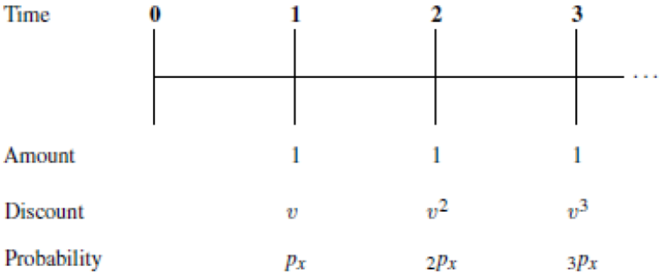

Figure 5.3 Time-line diagram for whole life immediate annuity.

Annual life annuities: immediate life annuities

- Consider a **n-year term immediate annuity** underwritten to (x) at time 0. It pays 1 at times $1, 2, \ldots, n$, provided (x) is alive.
- Benefit cash flow:

$$\sum_{t=1}^{\min(K_x,n)} (1, t) = \sum_{t=1}^{n} \left(1_{\{T_x>t\}}, t\right)$$

- Present value:

$$Y^* = \sum_{t=1}^{n} v^t \; 1_{\{T_x>t\}}$$

- Actuarial value:

$$\boxed{a_{x:\overline{n}|} \overset{\text{not.}}{=} \mathbb{E}\left[Y^*\right] = \textstyle\sum_{t=1}^{n} v^t \;\, {}_t p_x} \qquad (5.11)$$

- Relation:

$$\boxed{a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \;\, {}_n p_x} \qquad (5.12)$$

Annual life annuities: immediate life annuities

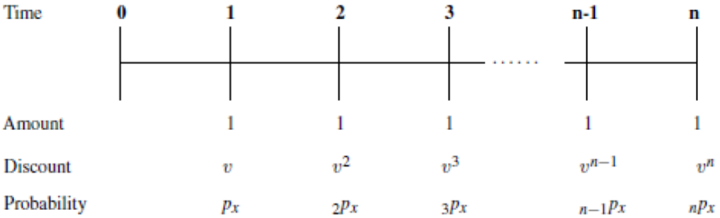

Figure 5.4 Time-line diagram for term life immediate annuity.

Deferred annuities

- Consider an annuity underwritten to (x) at time 0, with lifelong anual payments of 1 in advance, commencing at age $x + u$ (u is a non-negative integer).
- Benefit cash flow:

$$\sum_{k=u}^{K_x} (1, k)$$

- Actuarial value:

$$\boxed{{}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|}} \tag{5.25}$$

- Relation via actuarial discounting:

$${}_{u|}\ddot{a}_x = {}_uE_x \; \ddot{a}_{x+u} \tag{5.26}$$

Deferred annuities

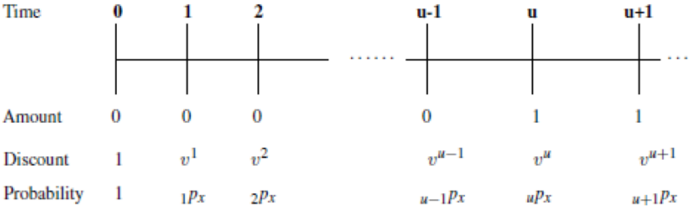

Figure 5.8 Time-line diagram for deferred annual annuity-due.