MTH6127 **Metric Spaces and Topology**

Course work 1 **26 January 2024**

1. For each of the following subsets of $\mathbb{R}$, determine the supremum and decide whether it is a maximum. Justify your answers.

   (a) $\{x : x^2 - 2 < 0\}$,

   (b) $\{x^2 - 2 : -2 \leq x < 2\}$,

   (c) $\{1 - 1/n^2 : n = 1, 2, 3, \ldots\}$ and

   (d) $\{1 + 1/n^3 : n = 1, 2, 3, \ldots\}$.

2. Let $X = \{0, 1\}^\omega$ be the set of all infinite sequences formed of 0s and 1s. For $x, y \in X$, define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|,$$

   where $x = (x_n)$ and $y = (y_n)$. Prove that d is well-defined and is a metric on X.

3. Prove that if (X, d) is a metric space then so also is (X, σ), where

$$\sigma(x, y) = \min\{d(x, y), 1\}.$$

4. A metric space X is said to be bounded if there is some number $M > 0$ such that

$$d(x, y) \leq M$$

   for any $x, y \in X$. Show that for any metric space (X, d), the metric space (X, σ) (as defined in the previous question) is bounded. Show also that the metric of example 2 is bounded.

5. The Euclidean norm of a vector $p = (p_1, p_2) \in \mathbb{R}^2$ is defined as $\|p\|_2 = \sqrt{p_1^2 + p_2^2}$. For $p, q \in \mathbb{R}^2$ define $d(p, q)$ by

$$d(p, q) = \begin{cases} 0, & \text{if } p = q; \\ \|p\|_2 + \|q\|_2, & \text{otherwise.} \end{cases}$$

   Prove that d is a metric on $\mathbb{R}^2$.