Connected Spaces

<u>Definition:</u> Let X be a top. space. We say that X is connected if any continuous map $f: X \Longrightarrow \{0,1\}$ is constant.

Here $\{0,1\} \subset \mathbb{R}$ has the induced topology.

2) Equivalently, X is connected if there is no surjective continuous map $f: X \to \{0,1\}$.

3) Equivalently, X is continuous if any subset $U \subset X$ which is both open and closed is either $\emptyset$ or X.

Indeed, if $U \subset X$, $U \neq \emptyset$, $U \neq X$ is open and closed then we may define

$$f: X \to \{0,1\}$$

by

$$f(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \notin U. \end{cases}$$

Since U and U^c are closed this map is continuous. f is not constant since $U \neq \emptyset$, $U \neq X$.

A set $U \subset X$ which is both closed and open is called <u>clopen</u>.

4) Equivalently, X is ~~continuous~~ connected if any cont. function $f: X \to \{0,1\}$ is constant

Examples: 1) $[a,b]$ is connected. Indeed there are no continuous surjective maps $f: [a,b] \longrightarrow \{0,1\} \subset \mathbb{R}$ by the mean value theorem.

2) Let $X = A \cup B$ where $A \cap B \neq \emptyset$. Assume that A and B are connected with respect to the induced topology. Then X is connected.

Proof: Let $f: X \longrightarrow \{0,1\}$ be a continuous fuction. Then $f|_A$ is constant and $f|_B$ is constant and since $A \cap B \neq \emptyset$ we see that f is onstant.

3) ~~Corollary:~~ $\mathbb{R}$ is connected
(a,b) is connected
$(a,b]$ is connected.

4) $\{0,1\}$ is disconnected.

5) $\mathbb{R} - \{0\}$ is disconnected:

$U = (-\infty, 0)$ is open and closed.

6) $\mathbb{Q}$ is disconnected:

$U = (-\infty, \sqrt{2}) \cap \mathbb{Q}$ is open & closed.

<u>Theorem:</u> Let $A \subset X$ be a connected subspace. If $A \subset B \subset \bar{A}$ then B is also connected.

<u>Proof:</u> Let $f: B \to \{0,1\}$ be cont. map. Then $f|_A$ is constant, say, $f|_A \equiv 0$. Then $U = f^{-1}(\{0\}) \subset B$

is an ~~open~~ clopen subset containing A, hence $U \supset B$, i.e. $U = B$ i.e. $f \equiv 0$.

<u>Theorem</u> The image of a connected space under a continuous map is connected.

<u>Proof:</u> Let $f: X \to Y$ be a surjective cont. map, X is connected. If

$$g: Y \to \{0,1\}$$

is surjective and continuous then

$$g \circ f: X \to \{0,1\}$$

is surj and cont $\Longrightarrow$

$$g \circ f \equiv \text{const}$$

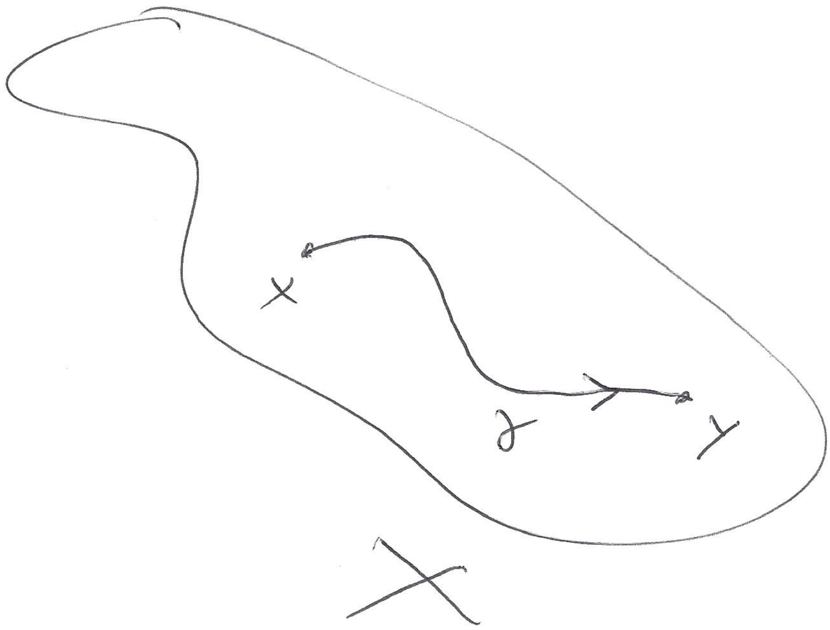
$$\Longrightarrow \quad g \equiv \text{const}. \qquad \Box$$

<u>Path-connected spaces</u>

X is path-connected if for any two points $x, y \in X$ there exists a cont.

$\gamma : [0,1] \to X$ with

$\gamma(0) = x, \quad \gamma(1) = y.$



Theorem: Any path-connected space is connected.

Proof: Let X be path-connected. Suppose that X is disconnected, i.e. $\exists$

$$f : X \to \{0,1\}$$

cont. & surjective.

$\exists x \in X,\ y \in X,\ f(x) = 0,\ f(y) = 1.$

Find $\gamma : [0,1] \to X$, $\gamma(0) = x$, $\gamma(1) = y$

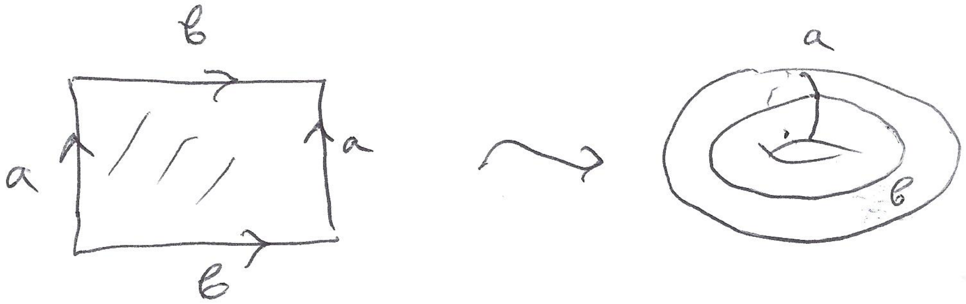
$$[0,1] \xrightarrow{\gamma} X \xrightarrow{f} \{0,1\}$$

is a cont. map which is not constant $0 \to 0$, $1 \to 1$.
We obtain a contradiction with the fact that $[0,1]$ is connected.

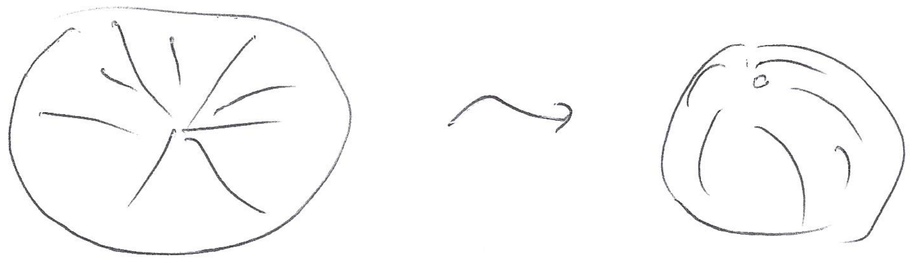
Quotient topology

[0,1]

S^1
circle.



T^2
torus.



D^2 - disk

S^2
sphere

Definition: Let X and Y be topological spaces. A surjective continuous map

$$p: X \to Y$$

is a quotient map if $U \subset Y$ is open if and only if $p^{-1}(U) \subset X$ is open.

Theorem: Let $f: X \to Y$ be a surjective continuous map. If X is compact and Y is Hausdorff then f is a quotient map.

Proof: Let $V \subset Y$ be such that $f^{-1}(V) \subset X$ is open. We want to show that $V \subset Y$ is open. Consider

$$\left(f^{-1}(V)\right)^c = f^{-1}(V^c) = A \subset X.$$

It is a closed subset of X. Since X is compact, A is compact.

$\Longrightarrow$ $f(A) = V^c \subset Y$ is compact.

Since Y is Hausdorff $f(A)$ is closed in Y, hence V^c is closed, i.e. $V \subset Y$ is open.

$\square$

<u>Definition:</u> If X is a topological space and A is a set and if

$$p: X \to A$$

is a surjective map, then there exists exactly one topology on A relative to which p is a quotient map.

$U \subset A$ is open $\iff$ $p^{-1}(U) \subset X$ is open.

$$p^{-1}\Big(\bigcup_{\alpha \in J} U_\alpha\Big) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha)$$

$$p^{-1}\Big(\bigcap_{i=1}^{n} U_i\Big) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$

——— " ———

<u>Quotient space:</u>

X – top. space

X^* – partition of X into disjoint subsets whose union is X.

$$p: X \longrightarrow X^*$$

Take the quotient topology on X^*.

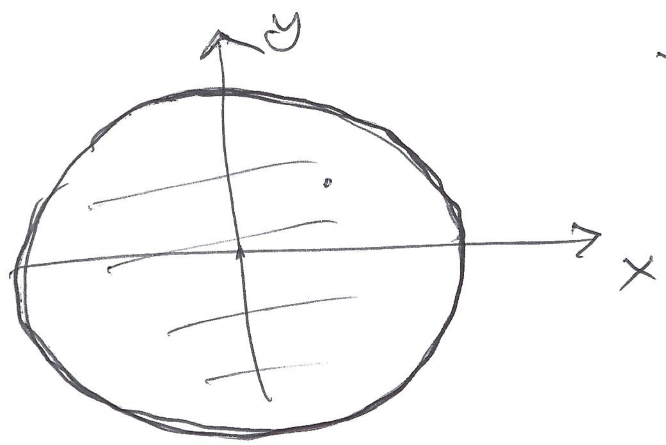
X^* – identification space.

Example: $X = D^2 \subset \mathbb{R}^2$

$D^2 = \{(x,y) ;\ x^2+y^2 \leq 1\}$.

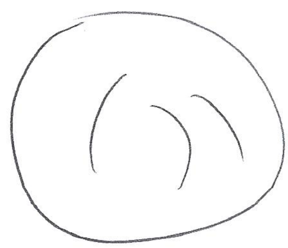
X^* – single points $\{(x,y)\}$ for $x^2+y^2<1$

and one set $\{(x,y);\ x^2+y^2=1\}$.



The boundary of D^2 becomes a single point.

$X/\sim$ the quotient is homeomorphic to S^2



. $S^2 \subset \mathbb{R}^3$

Maps of the quotient

Lemma: Let $p: X \to Y$ be a quotient map, where X and Y are topological spaces. Let Z be a topological space and let $f: Y \to Z$ be a map.

Then f is continuous if and only if $f \circ p : X \to Z$ is continuous.

<u>Proof:</u>

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "f\circ p"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

If f is continuous $\Longrightarrow$ $f \circ p$ is continuous as composition of two cont. maps.

Suppose that $f \circ p : X \to Z$ is cont. Let $U \subset Z$ be open. Then

$$(f\circ p)^{-1}(U) = p^{-1}\left(f^{-1}(U)\right) \subset X$$

is open $\Longrightarrow$ $f^{-1}(U) \subset Y$ is open (by the definition of quotient topology).

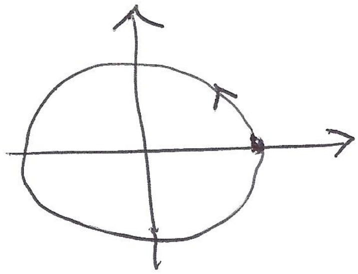
Hence f is continuous.

$\square$

<u>Universal property</u> of the quotient topology.

Example:

1) $f : [0,1] \longrightarrow S^1$, $\quad S^1 = \{ z \in \mathbb{C} ; |z| = 1 \}$.



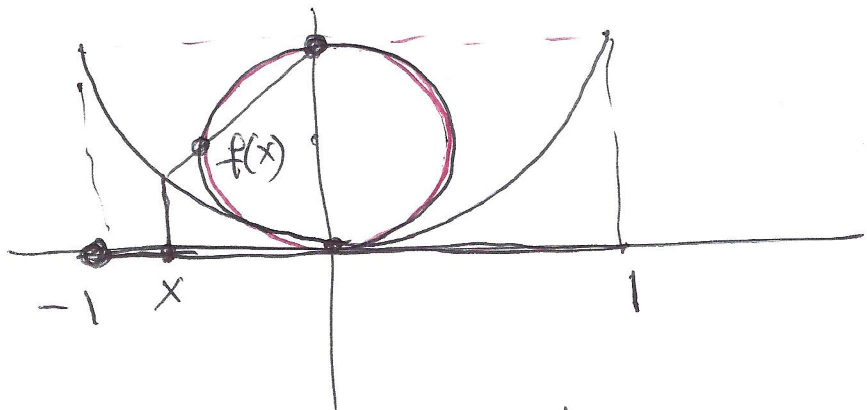
$f(t) = e^{2\pi i t}$,

$t \in [0,1]$.

f is continuous
f is surjective
$[0,1]$ is compact
S^1 is Hausdorff

Conclusion: f is a quotient map.

2) Stereographic projection

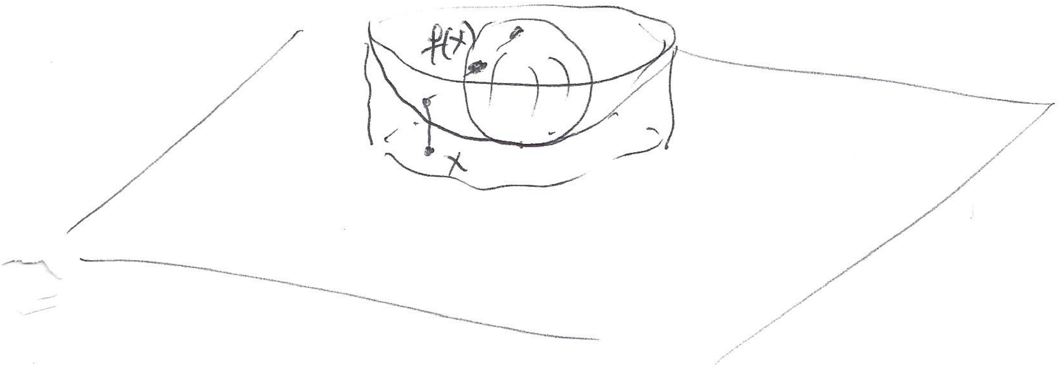


$f : [-1,1] \longrightarrow S^1$.

3) Stereographic projection

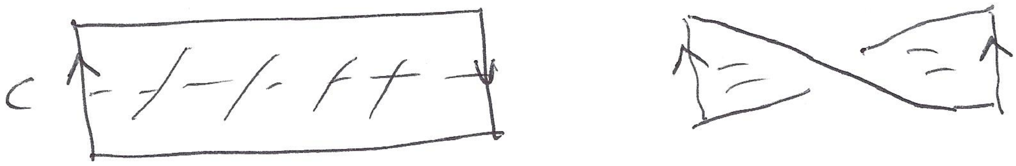
$$f : D^2 \longrightarrow S^2$$

$$f(\partial D^2) = * \in S^2 = \text{the North Pole.}$$



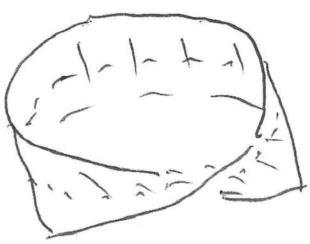
Some other quotient spaces

1. Möbius band M:

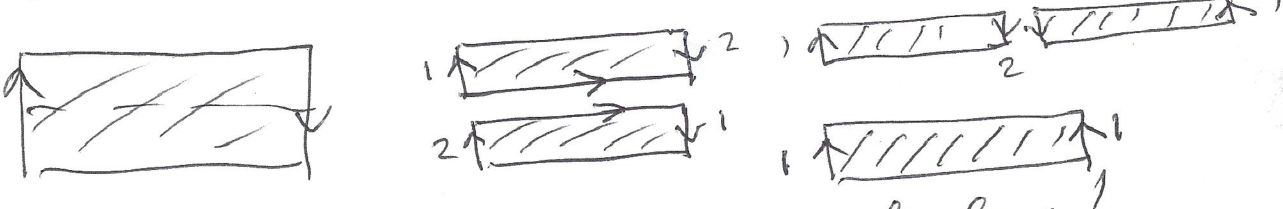


One-sided surface.

M

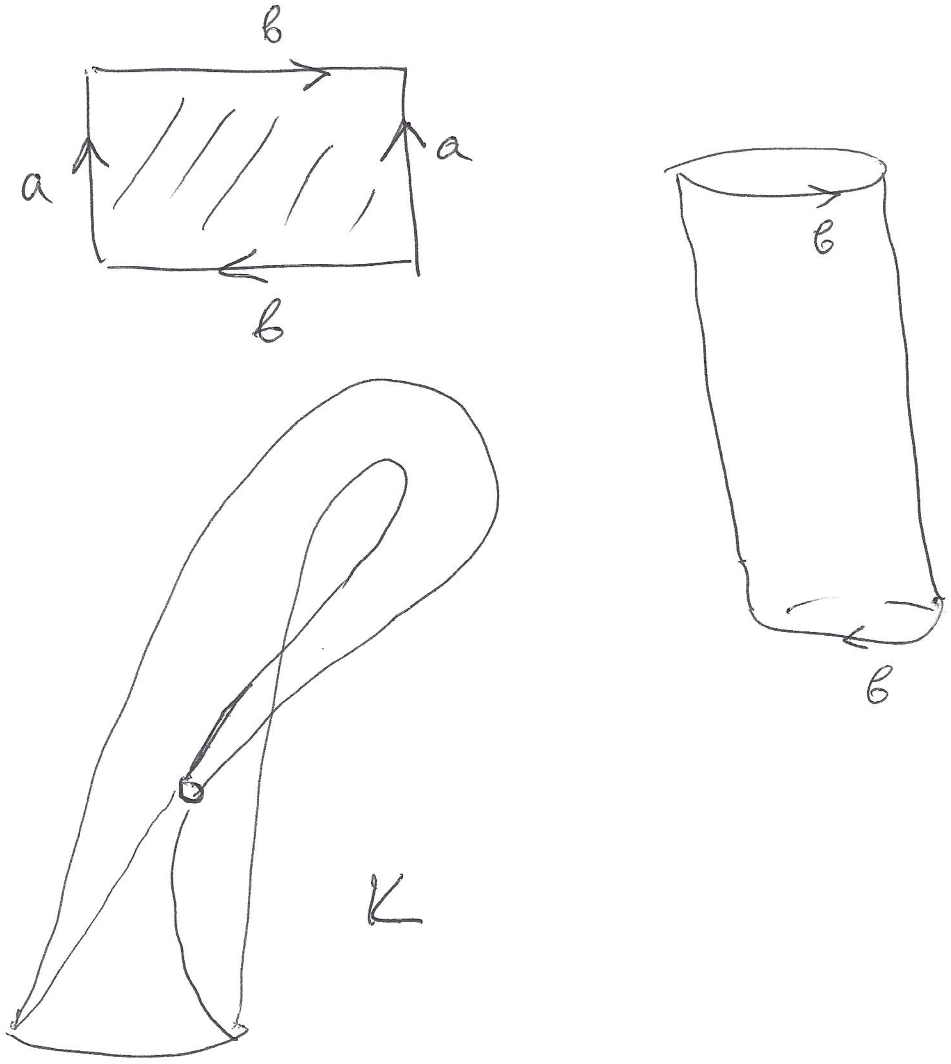


$C \subset M$ central line (circle).

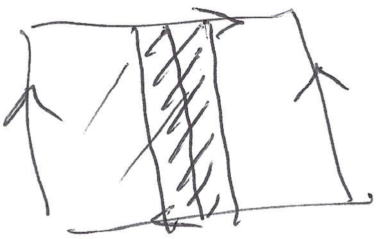


cylinder!

2) Klein bottle K.

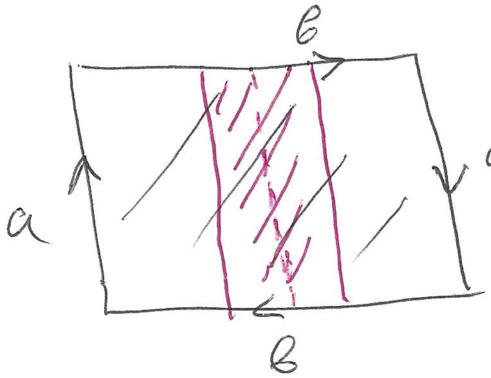


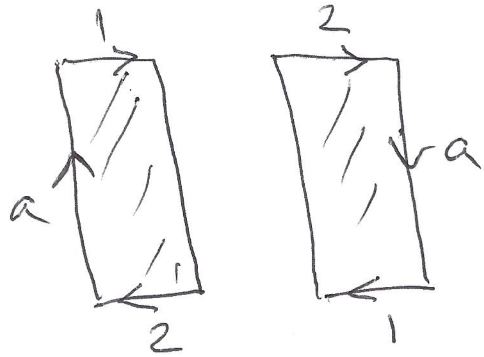
K contains the Möbius band M:



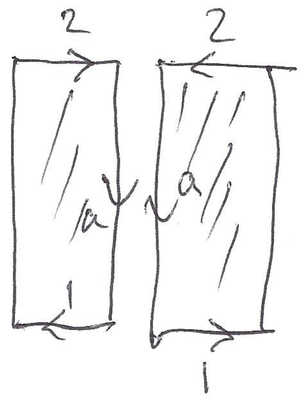
$K = M \cup M'$, $\quad M \cap M' = \partial M = \partial M'$

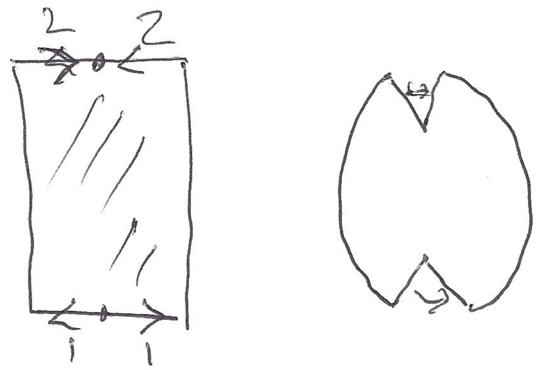
3) Projective plane P^2

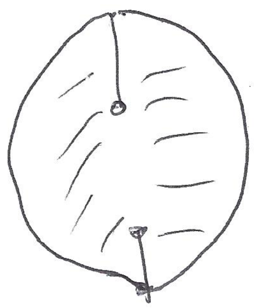




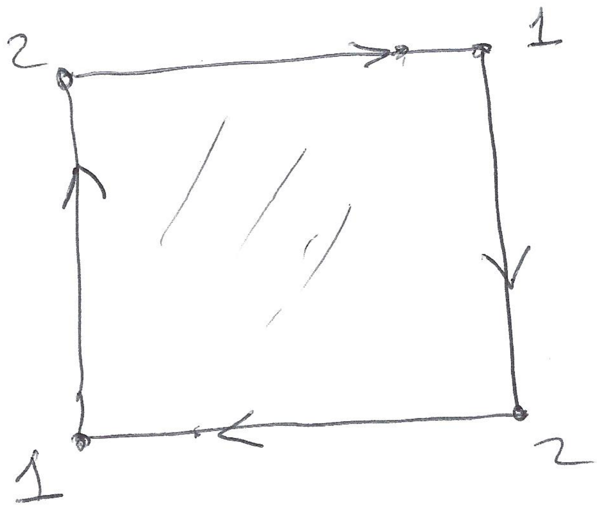
$$P^2 = M \cup D^2$$

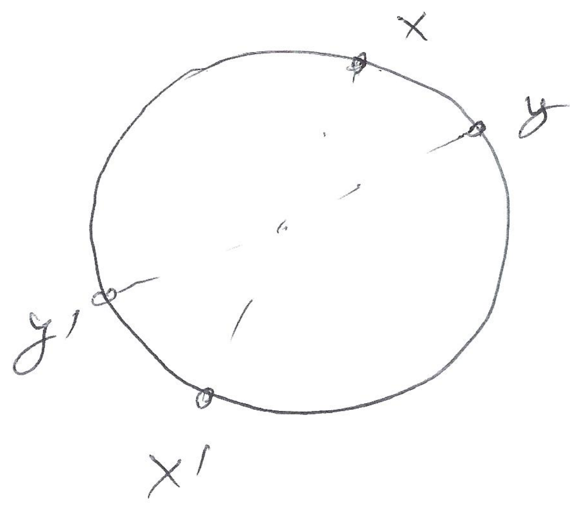




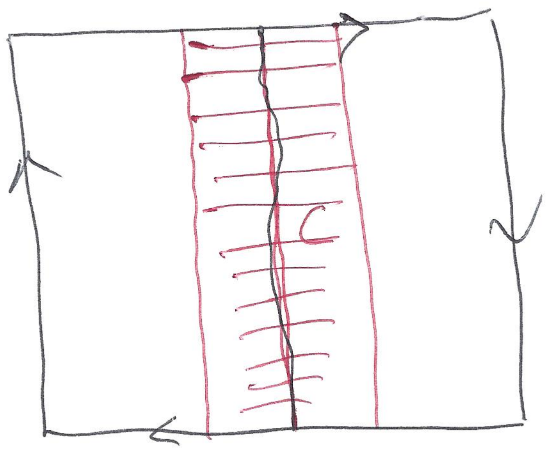


The real projective plane P^2

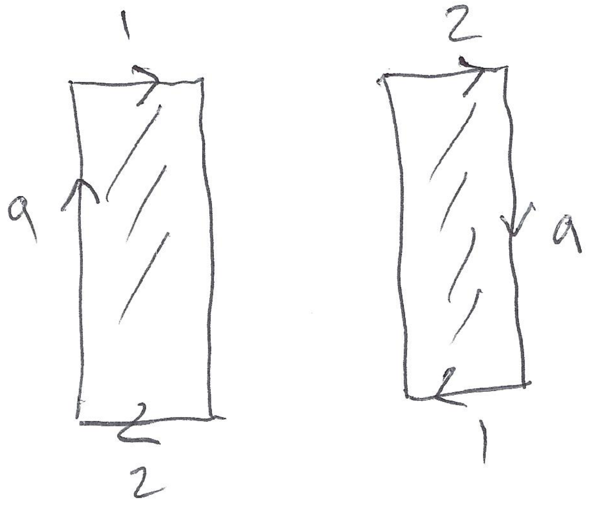


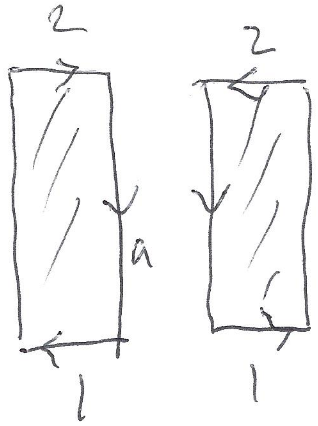


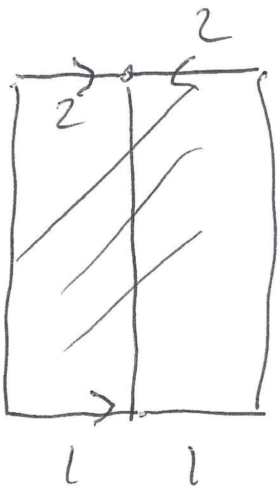
Decomposition of P^2

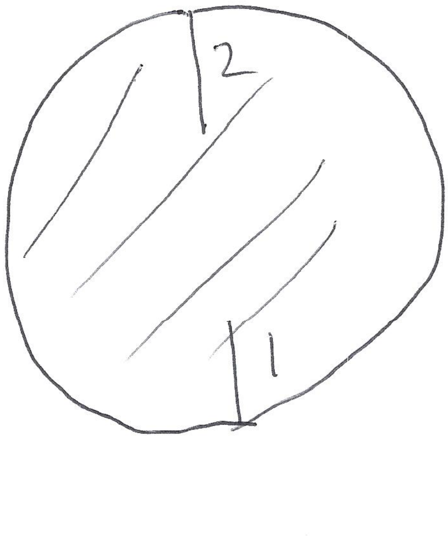


M









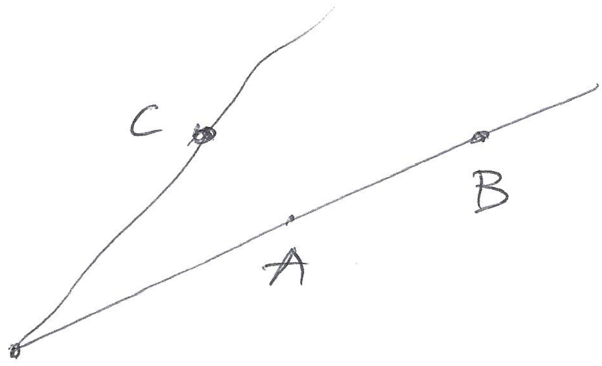
D^2

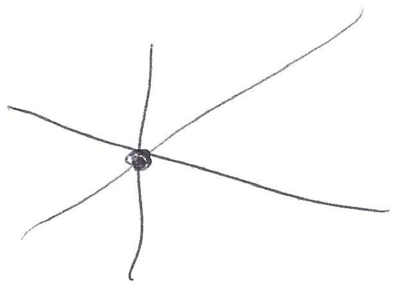
2-dimensional disc.

Conclusion: The real projective plane P^2 is the union $P = M \cup D^2$ of the Möbius band M and the disc D^2 such that

$$M \cap D^2 = \partial M = \partial D^2 \quad \text{– circle.}$$

Projective geometry





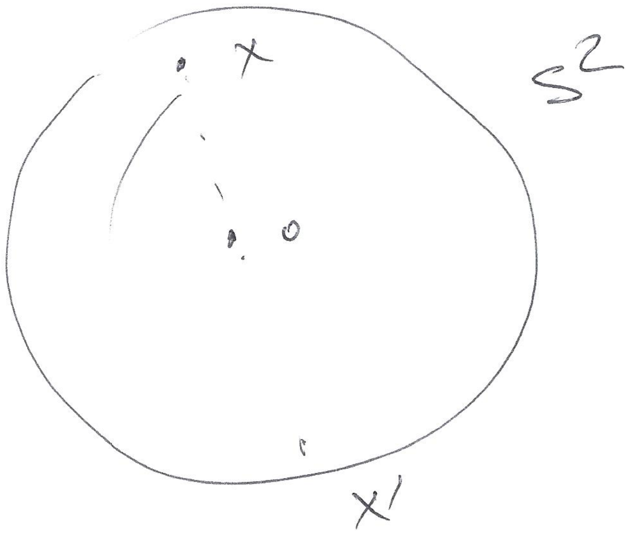
one eye

affine geometry

Π

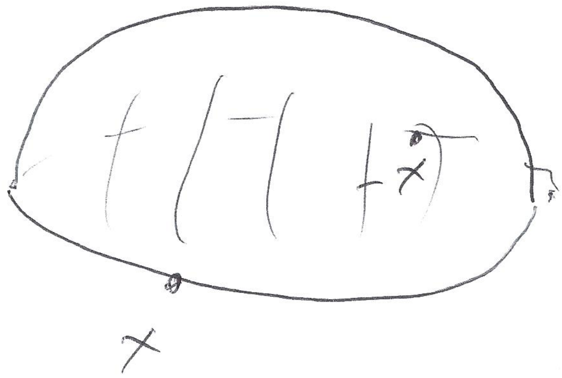
lines parallel to Π – points at ∞.
Circle of points at infinity.

<u>Definition</u>: The real projective plane is the quotient space $\mathbb{R}^3 - \{0\} / \sim$ where $x, y \in \mathbb{R}^3 - \{0\}$ are equivalent if $\exists \lambda \in \mathbb{R}$ such that $x = \lambda y$, i.e. x and y determine the same line through the origin.



<u>Definition</u>: The real projective plane P^2 is the quotient space $S^2/\sim$ where $x \sim -x$, $x \in S^2$.

~~Sher~~ Sphere S^2 with antipodal points identified.



upper semi-sphere.

Definition: The real projective plane P^2 is the quotient space $D^2/\sim$ where $x \sim -x$ for $x \in \partial D^2$.