MTH6127 **Metric Spaces and Topology**

Course work 10 **29 March 2024**

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1. Show that in the finite complement topology on $\mathbb{R}$ every subspace of $\mathbb{R}$ is compact.

2. Which of the following subsets of the real line $\mathbb{R}$ are compact; briefly explain your answer:

   (a) $(0, 1]$;

   (b) $[0, 1)$;

   (c) The Cantor set $F \subset [0, 1]$;

   (d) $[0, \infty)$;

   (e) $\mathbb{R} - \{0\}$.

3. Let $f : X \to Y$ be a continuous map between metric spaces. Show that if X is compact then for any closed subset $F \subset X$ the image $f(F) \subset Y$ is closed.

4. For $p \in [1, \infty)$ consider the space ℓ_p of all infinite sequences $(x_1, x_2, \dots)$ of real numbers satisfying $\sum_{n \geq 1} |x_n|^p < \infty$.

   (a) Show that the unit ball $B[0; 1] \subset \ell_p$ is not compact.

   (b) Show that $B[0; 1] \subset \ell_p$ is closed and bounded.

5. Let (X, d) be a metric space; let A be a non-empty subset of X. For each $x \in X$ define the distance from x to A by the equation

$$d(x, A) = \inf\{d(x, a); a \in A\}.$$

   (a) Show that $x \mapsto d(x, A)$ is a continuous function of x.

   (b) Show that if $A \subset X$ is compact then there exists $a \in A$ with $d(x, A) = d(x, a)$.

   (c) Show that if $A \subset X$ is closed and $x \notin A$ then $d(x, A) > 0$.

   (d) Define the ϵ-neighbourhood of A in X to be the set

$$U(A, \epsilon) = \{x; d(x, A) < \epsilon\}.$$

   Show that $U(A, \epsilon)$ equals the union of open balls $B(a, \epsilon)$ for $a \in A$.

   (e) Assume that $A \subset X$ is compact and $U \subset X$ is an open set containing A. Show that U contains some ϵ-neighbourhood of A.

   (f) Is the previous statement true if $A \subset X$ is closed but non-compact?