<u>Theorem:</u> Let X be a compact topological space. Then any closed subset $A \subset X$ is compact with the induced topology.

<u>Proof:</u> Let $\mathcal{V} = \{V_i\}_{i \in I}$ be an open cover of A. Each $V_i \subset A$ is open, i.e. $\exists$ an open subset $U_i \subset X$ such that

$$V_i = U_i \cap A .$$

Now, the family

$$\{U_i\}_{i \in I} \cup (X - A)$$

is an open cover of X. Since X compact it must have a finite subcover. Thus, for some finite

$$I' \subset I$$

the family

$$\{U_i\}_{i \in I'} \cup (X - A)$$

covers X. Then

$$\{V_i\}_{i \in I'} \cup (X - A)$$

is ~~or~~ a finite subcover of $\mathcal{V}$. I.e. A is compact. $\square$

Theorem: Let $f : X \to Y$ be a continuous surjective map. If X is compact then Y is compact as well.

Proof: Let $\mathcal{U} = \{V_i\}_{i \in I}$ be an open cover of Y. Then

$$\{f^{-1}(V_i)\}_{i \in I}$$

is an open cover of X. Since X is compact, there exists a finite subset

$$I' \subset I$$

such that

$$\{f^{-1}(V_i)\}_{i \in I'}$$

covers X. Then

$$\{V_i\}_{i \in I'}$$

covers Y (here we use the assumption that f is surjective).

Thus, Y is compact.

Corollary: Let $f: X \to Y$ be a homeomorphism. Then X is compact if and only if Y is compact.

Proof: If X is compact then $Y = f(X)$ is compact as image of a compact set.

If Y is compact we apply this argument to the inverse map
$$f^{-1}: Y \to X$$
which is continuous and surjective. $\square$

Example:

$(0,1)$ is homeomorphic to $\mathbb{R}$.

$\mathbb{R}$ is not compact

Hence $(0,1)$ is not compact.

Any open interval (a,b) is homeomorphic to $(0,1)$ and is not compact.

Finite intersection property (FIP)

Let $\{F_i\}_{i\in I}$ be a family of subsets of X. We say that $\{F_i\}_{i\in I}$ has the finite intersection property if for any finite subset $I'\subset I$ the intersection

$$\bigcap_{i\in I'} F_i \neq \emptyset .$$

Theorem: X is compact if and only if any family of closed subsets $F_i\subset X$ with finite intersection property has a non-empty intersection

$$\bigcap_{i\in I} F_i \neq \emptyset .$$

Proof: Suppose that X is compact and $\{F_i\}_{i\in I}$ is a family of closed subsets with FIP. Consider

$$U_i = F_i^{\,c} \qquad \text{(the complement)}$$

Then

$$\bigcup_{i\in I'} U_i \neq X \quad \text{for any finite subset } I'\subset I$$

Hence $\bigcup_{i\in I} U_i \neq X \Longrightarrow \exists\, \xi \in \bigcap_{i\in I} F_i .$

suppose that any family of closed sets with FIP has a non-empty intersection. Let

$\{U_i\}_{i\in I}$ be an open cover of X.

Define $F_i = U_i^c$.

Then $\bigcap_{i\in I} F_i = \emptyset$ (since U_i cover)

Then $\exists$ finite $I' \subset I$ with

$$\bigcap_{i\in I'} F_i = \emptyset.$$

Then $\bigcup_{i\in I'} U_i = X$, i.e. the original cover has a finite subcover.

$\dashv$

Bolzano - Weierstrass property

Theorem: Any infinite subset $A \subset X$ of a compact space has a limit point.

Recall: a point $x \in X$ is a limit point of A if any open subset $U \subset X$ containing x contains a point of $A - \{x\}$.

Proof: Suppose that A has no limit points. Hence for any point $x \in X$ there exist an open subset

$$x \in U_x \subset X$$

such that

$$U_x \cap A = \begin{cases} \emptyset & \text{or} \\ \{x\} & . \end{cases}$$

The family $\{U_x\}_{x \in X}$ is an open cover of X. Since X compact, it must have a finite subcover, i.e. $\exists\, x_1, \dots, x_N \in X$ such that

$$\bigcup_{i=1}^{N} U_{x_i} = X.$$

But each set V_{x_i} contais at most one point of A. Thus A mus have at most N points – contradiction.

$\square$.

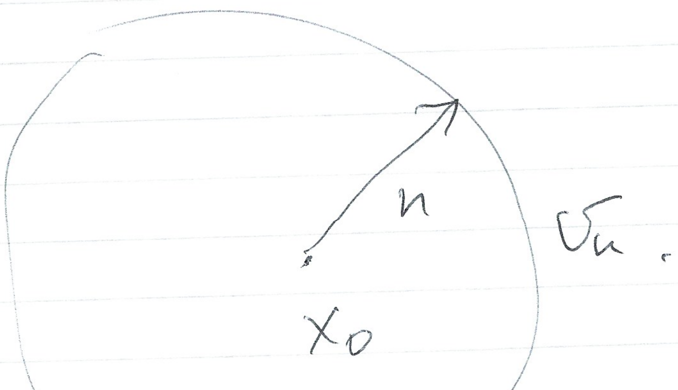
Theorem: In a metric space (X,d), every compact subset $A \subseteq X$ is bounded.

We say that $A \subset X$ is bounded if

$$\sup\{d(a,a') \; ; \; a,a' \in A\} < \infty.$$

Proof: Fix a point $x_0 \in A$ and consider the open balls

$$U_n = B(x_0, n) \subset X$$



The family $\{U_n\}_{n \geq 1}$, is an open cover of X.

The family

$$\mathcal{V} = \{ U_n \cap A ;\ n = 1, 2, \dots \}$$

is an open cover of A. Since A is compact, $\mathcal{V}$ has a finite subcover

$$\{ U_n \cap A ;\ n = n_1, \dots, n_N \}$$

Then $U_m \cap A = A$

where $m = \max\{ n_1, \dots, n_N \}$.

This means that

$$A \subset U_m$$

i.e. $d(a, x_0) < m \quad \forall a \in A$.

Therefore, for $a, a' \in A$ one has

$$d(a, a') \leq d(a, x_0) + d(a', x_0) < 2m.$$

We see that A is bounded.

$\square$

Bolzano - Weierstrass theorem II

Theorem: In a compact metric space X, any sequence $x_1, x_2, \dots$ has a convergent subsequence.

Proof: Consider the set

$$A = \{x_1, x_2, \dots \} \subset X$$

of points which appear in the sequence $x_1, x_2, \dots$.

Firstly, suppose that A is finite. Then at least one of its points is repeated infinitely many times. Then there exists a sequence of integers

$$n_1 < n_2 < n_3 < \dots$$

such that

$$x_{n_1} = x_{n_2} = x_{n_3} = \dots$$

We obtain a stationary subsequence $\{x_{n_k}\}_k$ which clearly converges.

Secondly, suppose that the set A is infinite. Then by the Bolzano - Weierstrass thm, A has a limit point, $x_0 \in X$. Then for any integer $k \geq 0$ the intersection

$$B(x_0, \tfrac{1}{k}) \cap \quad \neq \emptyset$$

is non-empty, i.e. we may find

$$x_{n_k} \in B(x_0, \tfrac{1}{k}) \cap A.$$

We obtain a subsequence

$$x_{n_1}, x_{n_2}, \dots$$

of the original sequence, and

$$d(x_{n_k}, x_0) < \frac{1}{k}.$$

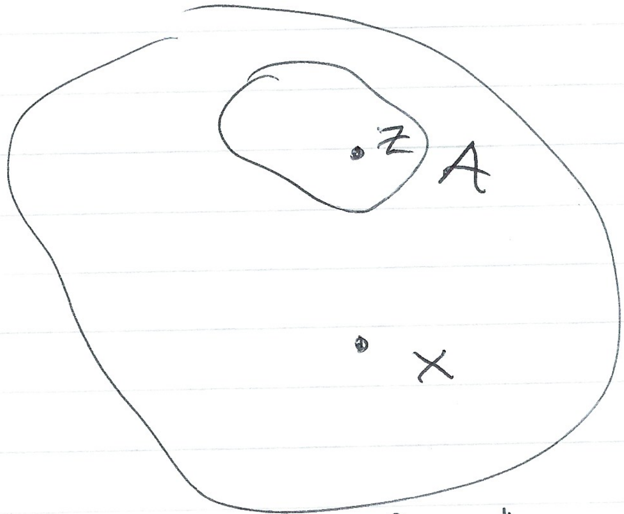
Then

$$x_{n_k} \longrightarrow x_0$$

$\square$

Theorem: If A is a compact subset of a Hausdorff space X, and if $x \in X - A$, then there exist disjoint neighbourhoods of x and A. Therefore a compact subset of a Hausdorff space is closed.

Proof:



Let $z \in A$ be a point of A. Since X is Hausdorff, $\exists$

$$U_z, V_z \subset X, \quad x \in U_z, \quad z \in V_z$$

and $$U_z \cap V_z = \emptyset.$$

We obtain an open cover

$$\{V_z\}_{z \in A}$$

of A. Since A is compact, $\exists$ finitely many points $z_1, \ldots, z_N$ with

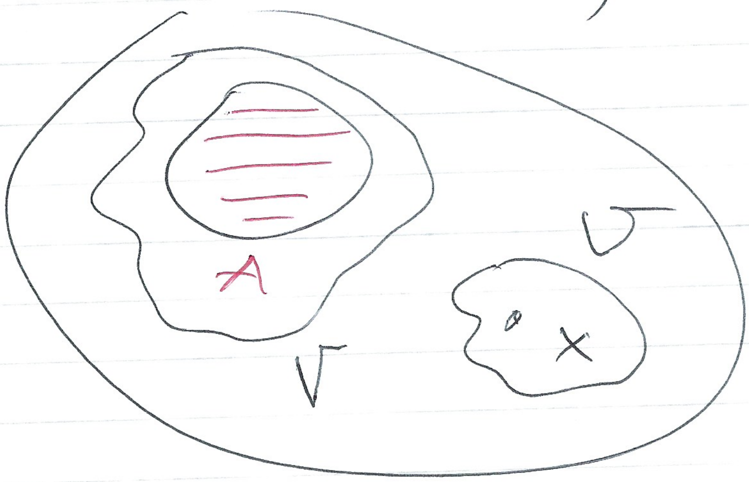
$$\bigcup_{i=1}^{N} V_{z_i} \supset A.$$

Then

$$V = \bigcup_{i=1}^{N} V_{z_i} \quad \text{and} \quad U = \bigcap_{i=1}^{N} U_{z_i}$$

are disjoint, open, and

$$V \supset A \,, \quad U \ni z \,.$$



A is closed since $X - A$ is open:
any $x \in X - A$ has an open nbd $U \subset X - A$.

$\square$

Example.

Let $X = \ell_2$ - the space of sequences $(x_1, x_2, \dots)$ with

$$\sum |x_i|^2 < \infty$$

Consider the closed unit ball

$A = B[0, 1] \subset \ell_2$.

$$A = \{ (x_1, x_2, \dots); \ \sum_i |x_i|^2 \le 1 \}.$$

A is closed & bounded but it is not compact.

Indeed, consider the sequence

$\{e_n\}$, $e_n = (0, \dots 0, \underset{n}{1}, 0 \dots) \in \ell_2$.

This sequence is not Cauchy:

$$d(e_n, e_m) = \sqrt{2}$$

for $n \ne m$. ~~The sequen~~ The set

$$S = \{e_1, e_2, \dots \} \subset B[0, 1] = A$$

is an ~~s~~ infinite subset of a compact having no limit points violating the Bolzano-Weierstrass property.

Continuous functions on compact spaces

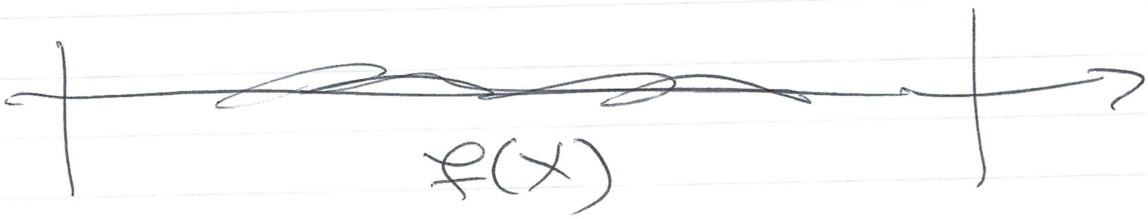
$f: X \to \mathbb{R}$, X - top. space
$\mathbb{R}$ - with the standrd topology.

Theorem: If X is compact then any cont. function

$$f: X \to \mathbb{R}$$

is bounded and attains its bounds.

Proof: $f(X) \subset \mathbb{R}$ is compact, i.e. it is closed and bounded. Thus f is bounded.



Since $f(X) \subset \mathbb{R}$ is closed, both supremum and infimum lie in $f(X)$. We can therefore find $x_1, x_2 \in X$ such that

$$f(x_1) = \sup(f(X))$$

$$f(x_2) = \inf(f(X)),$$

i.e. f attains its bounds. $\square$

<u>Theorem</u>: Let $f: X \to Y$ be a continuous bijective map. If X is compact and Y is Hausdorff then f is a homeomorphism.

<u>Proof</u>: We want to show that

$$f^{-1}: Y \to X$$

is continuous, i.e. for any closed $F \subset X$, the set

$$(f^{-1})^{-1}(F) = f(F) \subset Y$$

is closed.

- $F \subset X$ is closed & X is compact $\Longrightarrow$ F is compact.
- Hence $f(F)$ is compact.
- $f(F)$ compact & Y Hausdorff $\Longrightarrow$ $f(F) \subset Y$ is closed.

$\square$