MTH6127 **Metric Spaces and Topology**

Course work 9 **22 March 2024**

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1. Let X and Y be topological spaces and let $f : X \to Y$ be a map. Show that the following properties of f are equivalent:

   (a) f is continuous;

   (b) For every subset $A \subset X$ one has $f(\overline{A}) \subset \overline{f(A)}$.

   (c) For every closed set $F \subset Y$ the preimage $f^{-1}(F) \subset X$ is closed in X.

2. In the finite complement topology on $\mathbb{R}$, to what point (or points) does the sequence $x_n = 1/n$ converge?

3. Let $y_n = 1$ for n even and $y_n = -1$ for n odd. In the finite complement topology on $\mathbb{R}$, to what point (or points) does the sequence y_n converge?

4. Show that a subspace of a Hausdorff space is Hausdorff.

5. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by the equation

$$F(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if} \quad (x, y) \neq (0, 0), \\ 0, & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

   (a) Show that for any $x_0 \in \mathbb{R}$ the function $y \mapsto F(x_0, y)$ is continuous.

   (b) Show that for any $y_0 \in \mathbb{R}$ the function $x \mapsto F(x, y_0)$ is continuous.

   (c) Show that the function $x \mapsto F(x, x)$ is discontinuous.

   (d) Show that $F : \mathbb{R}^2 \to \mathbb{R}$ is discontinuous.

6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous at every point $x \in \mathbb{R}$.

7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at a single point $x \in \mathbb{R}$.