DEFINITION 7.16. A sequence of points x_n of a topological space $(X, \mathcal{T})$ converges to a point $x_0 \in X$ if for any open neighbourhood $x \in U \subset X$ there exists N such that for all $n > N$ one has $x_n \in U$.

This definition coincides with the definition of convergence in metric spaces. However in general topological spaces convergence of sequences has sometimes different unexpected properties.

EXAMPLE 7.17. Consider the real line $\mathbf{R}$ with the finite complement topology of Example 7.7. Let $x_n = n$ where $n = 1, 2, 3, \ldots$. Then $x_n \to x_0$ where $x_0 \in \mathbf{R}$ is arbitrary. Indeed, any open subset U containing x_0 contains all real numbers besides finitely many, hence it must contain all sufficiently large integers $n > N$. We see that in topological spaces (unlike the case of metric spaces) a sequence may have many limits.

7.3. Hausdorff and T_1-spaces

A topological space $(X, \mathcal{T})$ is said to be *Hausdorff* if any two distinct points $x, y \in X$, $x \neq y$, admit disjoint open neighbourhoods, i.e. open subsets $U, V \in \mathcal{T}$, $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Any metric space (X, d) is Hausdorff. Indeed, we can take $U = B(x; r)$ and $V = B(y; r)$ where $0 < 2r < \mathsf{d}(x, y)$.

We say that a topological space $(X, \mathcal{T})$ is a T_1-*space* if every single point set $\{x\}$ is closed. This means that for any pair of distinct points $x, y \in X$, $x \neq y$, there exists open subsets $U, V \subset X$ such that $x \in U$, $y \in V$ and $x \notin V$, $y \notin U$, see Figure 1.

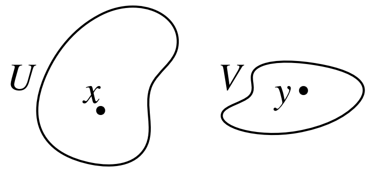


FIGURE 1. Disjoint neighbourhoods of points in a Hausdorff space

Any Hausdorff space satisfies the T_1-axiom.

PROPOSITION 7.18. *In a Hausdorff topological space $(X, \mathcal{T})$ a sequence of point may have at most one limit.*

PROOF. Suppose the contrary, i.e. a sequence of points $x_n \in X$ has two distinct limits, i.e. $x_n \to x_0$ and $x_n \to x_0'$ where $x_0 \neq x_0'$. Consider disjoint open neighbourhoods $x \in U$, $y \in V$, $U \cap V = \emptyset$. Then there exists N such that for all $n > N$ one has $x_n \in U$ and $x_n \in V$ which is impossible. $\square$

7.4. Continuous maps between topological spaces

The definition of continuous map between topological spaces is analogous to the corresponding definition for metric spaces although we use the language of open sets and avoid using metrics.

DEFINITION 7.19. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. A map $f : X \to Y$ is continuous at a point $x_0 \in X$ if for every neighbourhood $U \subset Y$ of $y_0 = f(x_0)$ there exists a neighbourhood $V \subset X$ of x_0 such that $f(V) \subset U$. We say that a map $f : X \to Y$ is continuous if it is continuous at every point $x_0 \in X$.

Note that for maps between metric spaces this notion of continuity coincides with the one we studied previously.

LEMMA 7.20. *Given topological spaces* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$*. The following properties of a map* $f : X \to Y$ *are equivalent:*

(i) f *is continuous;*
(ii) *for every open subset* $U \subset Y$ *the preimage* $f^{-1}(U) \subset X$ *is open.*
(iii) *for every closed subset* $F \subset Y$ *the preimage* $f^{-1}(F) \subset X$ *is closed.*

PROOF. Let us assume that f is continuous. Let $U \subset Y$ be an open subset. For any $x \in f^{-1}(U)$ the image $y = f(x)$ lies in U and by Definition 7.19 there is a neighbourhood $V_x \subset X$ of x such that $V_x \subset f^{-1}(U)$. We see that the set

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x$$

is the union of open sets and hence it is open. We have shown that (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii) is obvious since

for a closed $F \subset Y$ the set $Y - F$ is open and hence by (ii) the preimage $f^{-1}(Y - F)$ is open in X implying that its complement

$$f^{-1}(F) = X - f^{-1}(Y - F)$$

is closed in X.

Finally we show that (iii) $\Longrightarrow$ (i). Suppose that $f : X \to Y$ satisfies (iii). Let $x_0 \in X$ and let $U \subset Y$ be a neighbourhood of $y_0 = f(x_0)$. The complement $F = Y - U$ is closed and by (iii) the preimage $F' = f^{-1}(F) \subset X$ is closed. Since $x_0 \notin F'$ there is a neighbourhood $V \subset X$ of x_0 which is disjoint from F'. This means that $V \subset f^{-1}(U) = X - F'$. □

EXAMPLE 7.21. Consider the continuous function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$. The image of the open interval $(-1, 1)$ equals $[0, 1)$ which is not open.

Compact topological spaces.

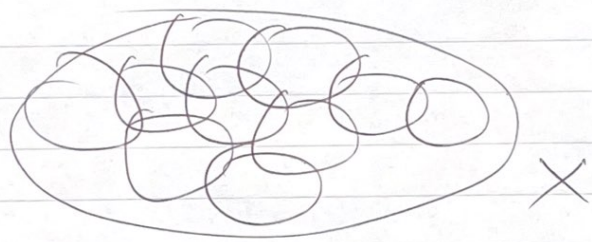
Let X be a topological space.
A family of open sets

$$\mathcal{U} = \{U_i\}_{i \in I}$$

is an open cover of X if

$$\bigcup_{i \in I} U_i = X$$

i.e. $\forall x \in X \;\; \exists i \in I$ such that $x \in U_i$.



A subcover of $\mathcal{U} = \{U_i\}_{i \in I}$ is any family

$$\mathcal{U}' = \{U_i\}_{i \in I'}$$

where $I' \subset I$, such that

$$\bigcup_{i \in I'} U_i = X.$$

<u>Definition:</u> We say that a topological space X is compact if any open cover of X has a finite subcover.

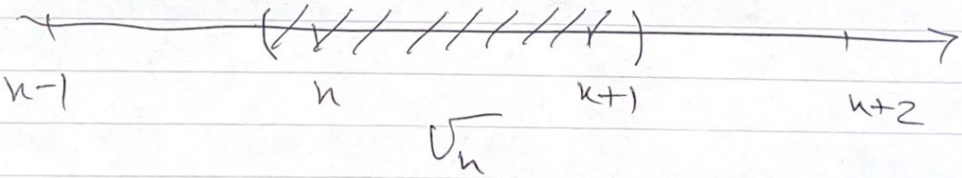
In other words, if $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X then for some finite subset $I' \subset I$ the family

$$\mathcal{U}' = \{U_i\}_{i \in I'}$$

is also a cover of X.

<u>Example:</u> Let $X = \mathbb{R}$ with the usual topology. Denote

$$U_n = \left(n - \tfrac{1}{4},\ n + 1 + \tfrac{1}{4}\right)$$

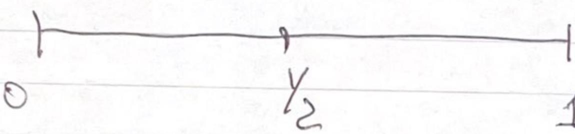


The family $\mathcal{U} = \{U_n\}_{n \in \mathbb{Z}}$ is an open cover of $\mathbb{R}$. But the point $n + \frac{1}{2}$ is covered by U_n only. Hence $\mathcal{U}$ has no proper subcovers. We see that $\mathbb{R}$ is <u>not</u> compact.

<u>Theorem (Heine-Borel)</u>: The closed interval $[0,1]$ is compact.

<u>Proof.</u> Suppose that $[0,1]$ is not compact, i.e. $\exists$ an open cover $\mathcal{U}$ which has no finite subcovers.

Consider the subdivision



At least one of the intervals $[0,\frac{1}{2}]$ or $[\frac{1}{2}, 1]$ cannot be covered by a finite subcover of $\mathcal{U}$.

Denote such subinterval $[a_1, b_1]$.

Next we subdivide $[a_1, b_1]$ into two equal size subintervals

$$[a_1, c] \text{ and } [c, b_1]$$

where $c = \frac{a_1+b_1}{2}$. At least one of $[a_1, c]$ or $[c, b_1]$ cannot be covered by a finite subcover of $\mathcal{U}$ since otherwise, if $[a_1, c]$ is covered by $\{U_i\}_{i\in I'}$ and $[c, b_1]$ is covered by $\{U_i\}_{i\in I''}$ where $I', I'' \subset I$ are finite, then $[a_1, b_1]$ would be covered by

$$\{U_i\}_{i\in I'\cup I''}$$

which is a finite subcover of $\mathcal{U}$. Denote such subinterval $[a_2, b_2]$.

Continuing similarly we obtain an infinite sequence of intervals

$$[a_0, b_0] = [0, 1], [a_1, b_1], [a_2, b_2], \ldots$$

such that:

1) $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$ "nested sequence"
2) $|b_n - a_n| = \frac{1}{2^n}$
3) Each $[a_n, b_n]$ has the property that no finite subfamily of $\mathcal{U}$ covers $[a_n, b_n]$.

Consider $\bigcap_{n=0}^{\infty} [a_n, b_n]$.

We claim that it is a single

$\mathcal{S}$

point set $\{\xi\}$. Indeed, we have

$$a_0 \le a_1 \le a_2 \le \dots \qquad \dots \le b_2 \le b_1 \le b_0$$

Let $\xi = \sup\{a_n\}$, $\xi' = \inf\{b_n\}$.

Then $\xi \in [a_n, b_n]$, $\xi' \in [a_n, b_n]$

for any n, and since

$|b_n - a_n| = \frac{1}{2^n}$ we see that

$$|\xi - \xi'| \le \frac{1}{2^n} \quad \forall n$$

Thus, $\xi = \xi'$.

Consider the point $\xi \in [0,1]$. We know that for some $i \in I$, $\xi \in U_i$. Since U_i is open, $\exists \varepsilon > 0$ such that

$$(\xi - \varepsilon, \xi + \varepsilon) \subset U_i.$$

Take n so large that $\frac{1}{2^n} < \varepsilon$.

Then $[a_n, b_n] \subset (\xi - \varepsilon, \xi + \varepsilon) \subset U_i$

We see that $[a_n, b_n]$ is covered by a single open set U_i of the cover $\mathcal{U}$. This contradicts the property 3) above.

Hence, $[0,1]$ is compact.

$\square$

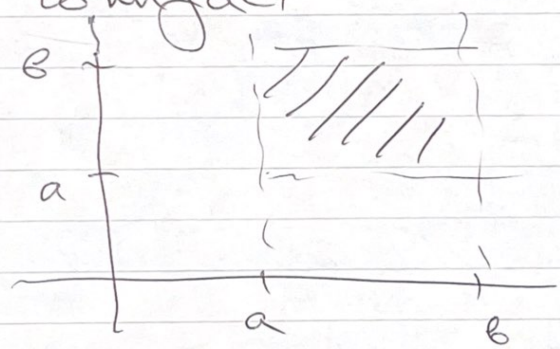
Similar arguments prove that:

- any closed interval $[a,b]$ is compact

- any square

$$[a,b]\times[a,b]\subset\mathbb{R}^2$$

is compact



- any n-dimensional cube

$[a,b]^n\subset\mathbb{R}^n$ is compact.

Here

$$[a,b]^n=\{(x_1,\dots,x_n)\in\mathbb{R}^n;\ \underset{i=1,\dots,n}{x_i\in[a,b]}\}$$

Note: this becomes false in infinite dimensional spaces.