EXAMPLE 6.4. Consider the equation

$$\frac{1}{2} \cdot \cos x + 5 = x. \tag{6.4}$$

Denoting $f(x) = \frac{1}{2} \cdot \cos x + 5$ we have $f : \mathbf{R} \to \mathbf{R}$ and $f'(x) = -\frac{1}{2} \sin x$. Thus,

$$|f'(x)| \leq \frac{1}{2},$$

i.e. $f : \mathbf{R} \to \mathbf{R}$ is a contraction mapping. Staring with $x_0 = 8$ we obtain
$x_1 = 4.927$,
$x_2 = 5.10$,
$x_3 = 5.19$.
$x_4 = 5.23$.
$x_5 = 5.24$,
$x_6 = 5.255$,
$x_7 = 5.258$,
$x_8 = 5.259$,
$x_9 = 5.26$,
$x_{10} = 5.26$.

6.2.2. Suppose we need to find a solution of the equation $F(x) = 0$ lying in the interval $[a, b]$, where $F(a) < 0$ and $F(b) > 0$.

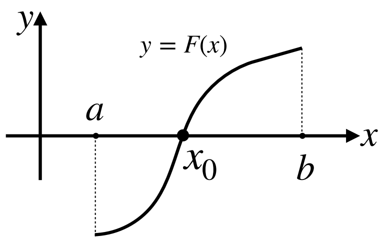

FIGURE 2. Equation $F(x) = 0$

Denote

$$f(x) = x - \lambda F(x), \quad \lambda > 0.$$

Then solutions of the equation $F(x) = 0$ are solutions of the fixed point equation $f(x) = x$ and vice versa. Suppose that

$$0 < K_1 \leq F'(x) \leq K_2 \quad \text{for} \quad x \in [a, b].$$

Then the derivative

$$f'(x) = 1 - \lambda F'(x)$$

satisfies

$$1 - \lambda K_2 \leq f'(x) \leq 1 - \lambda K_1.$$

We want

$$1 - \lambda K_1 < 1 \quad \text{and} \quad 1 - \lambda K_2 > -1.$$

The left inequality is automatically satisfied. The right inequality is equivalent to

$$\lambda < \frac{2}{K_2}. \tag{6.5}$$

For instance, one can take

$$\lambda = \frac{1}{K_2}$$

and solve the equation $F(x) = 0$ by the iterations

$$x_n = f(x_{n-1}), \quad x_n \to x_0.$$

EXAMPLE 6.5. We need to solve the equation $x^3 = 5$. The solution is $x = 5^{1/3} = 1.7099$.
To find this solution we may apply the method of iterations.
Let $F(x) = x^3 - 5$, $a = 1$, $b = 2$.
We have $F'(x) = 3x^2 > 0$, $K_1 = 3 \le F'(x) \le 12 = K_2$ for $x \in [1, 2]$.
Take $\lambda = 1/10 < 2/12 = 1/6$.
The equation

$$f(x) = x - \frac{1}{10}(x^3 - 5) = x + 1/2 - \frac{1}{10}x^3.$$

We obtain succesive iterations:
$x_0 = 1.5$
$x_1 = 1.66$
$x_2 = 1.702$
$x_3 = 1.7089$
$x_4 = 1.7098$.
We see that the iterations approach the solution 1.7099.

6.3. Theorem of Picard: Existence and uniqueness of solutions of differential equations

Consider a differential equation:

$$\frac{dy}{dx} = f(x, y(x)) \tag{6.6}$$

with the initial condition

$$y(x_0) = y_0. \tag{6.7}$$

We want to find a smooth function $y = y(x)$ satisfying the differential equation (6.6) as well as the initial condition (6.7). The following result is the classical Picard theorem which uses the contraction mappings and the Fixed Point Theorem 6.2 in its proof.

THEOREM 6.6. *Suppose that a real valued function $f(x, y)$ is defined on an open subset $G \subset \mathbf{R}^2$ containing the point (x_0, y_0). Moreover, assume that the function $f(x, y)$ is continuous, bounded $|f(x, y)| \le K$ and satisfies the Lipschitz condition with respect to y, i.e.*

$$|f(x, y_1) - f(x, y_2)| \le M|y_1 - y_2| \quad \text{where} \quad (x, y_1), (x, y_2) \in G. \tag{6.8}$$

Let $0 < \delta < M^{-1}$ be such that the rectangle $[x_0 - \delta, x_0 + \delta] \times [y_0 - K\delta, y_0 + K\delta]$ is contained in G. Then there exists a unique solution of the differential equation (6.6) on the interval $(x_0 - \delta, x_0 + \delta)$ satisfying the initial condition (6.7).

Proof. The equation (6.6) and the initial condition (6.7) can equivalently be presented in the form of an integral equation

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt. \tag{6.9}$$

In more detail, any function $\phi(x)$ satisfying (6.9) will automatically satisfy (6.7) and (6.6). The integral equation (6.9) is a fixed point equation for a certain contraction operator which we clarify below.

Denote by C^* the space of all continuous functions $\phi : [x_0 - \delta, x_0 + \delta] \to [y_0 - K\delta, y_0 + K\delta]$ with the metric

$$\mathsf{d}(\phi_1, \phi_2) = \sup_t |\phi_1(t) - \phi_2(t)| = \max_t |\phi_1(t) - \phi_2(t)|.$$

This space is complete as a closed subspace of the complete metric space $C[x_0 - \delta, x_0 + \delta]$.

Consider the mapping $A : C^* \to C^*$ given by $A\phi = \psi$ where

$$\psi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt.$$

We have $\psi \in C^*$ since for $|x - x_0| \leq \delta$ one has

$$|\psi(x) - y_0| = \left| \int_{x_0}^{x} f(t, \phi(t))dt \right| \leq K\delta.$$

The integral equation (6.9) can be written as $\phi = A\phi$, i.e. it is the fixed point equation for the operator A. The mapping A is a contraction. Indeed,

$$|\psi_1(x) - \psi_2(x)| \leq \int_{x_0}^{x} |f(t, \phi_1(t)) - f(t, \phi_2(t))| \, dt \leq (M\delta) \cdot \mathsf{d}(\phi_1, \phi_2),$$

implying that

$$\mathsf{d}(A\phi_1, A\phi_2) \leq (M\delta) \cdot \mathsf{d}(\phi_1, \phi_2), \quad \text{and} \quad M\delta < 1.$$

The Fixed Point Theorem 6.2 applies and gives the existence and uniqueness of a solution $\phi \in C^*$ to the equation (6.9). This completes the proof. □

CHAPTER 7

Topological Spaces and Continuous Functions

7.1. Topologies and Topological Spaces

Definition 7.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(T1) The empty set $\emptyset$ and the whole set X are in $\mathcal{T}$.
(T2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
(T3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a specified topology $\mathcal{T}$ on X is called *a topological space*. Sometimes we represent a topological space in the form of a pair $(X, \mathcal{T})$ to emphasise its topology. However in most cases we shall indicate the topological space by a single symbol X with understanding of which collection of subsets $\mathcal{T}$ forms its topology.

The subsets $U \in \mathcal{T}$ are said to be *open* or *open with respect to the topology* $\mathcal{T}$. We shall see that the same set X may carry many different topologies and thus it is important to specify which topology we have in mind.

We can rephrase Definition 7.1 as follows:

(T1) The empty set $\emptyset$ and X are open.
(T2) The union of any family of open sets is open.
(T3) The intersection of the elements of any finite collection of open sets is open.

Definition 7.2. Let $A \subset X$ be a subset of a topological space $(X, \mathcal{T})$. *A neighbourhood* of A is an open subset $U \subset X$ containing A.

We shall consider a few examples.

Example 7.3. Let (X, d) be a metric space. We defined the family of open subsets of X as the sets which can be represented as unions of open balls. This family of open sets has the properties (T1), (T2), (T3) and hence defines a topology on X. We shall refer to it as to *the metric topology*.

Metric topological spaces are of most importance in mathematics. Nevertheless, in many questions concerning the metric spaces it is more convenient to speak about their topology and forget about their metric. Moreover, sometimes it is more convenient to define topology rather than to define a metric. We shall see many such example in this course.

Example 7.4. The standard metric on $\mathbf{R}$ defines a topology which we shall refer to as *the standard topology*. The m-dimensional space $\mathbf{R}^m$ has also *the standard topology* which is generated by the Euclidean metric. It is easy to see that any metric d_p on $\mathbf{R}^m$, where $p \in [1, \infty]$, generates the standard topology; for this purpose one can use the inequalities (2.6).

Example 7.5. Suppose that $\emptyset$ and X are the only open sets. This is a topology on X which is called *indiscrete topology*. Such topology cannot be generated by a metric assuming that X contains at least two points.

EXAMPLE 7.6. Another extreme case is *the discrete topology* which declares any subsets of X to be open. The discrete topology is metrisable: it is generated by the metric $d(x, y) = 1$ for all $x \neq y$, $x, y \in X$.

EXAMPLE 7.7. Another interesting example is provided by *the finite complement topology.* In this topology nonempty open subsets $U \subset X$ are such that their complements $X - U$ are finite. Any union of sets with finite complements also has a finite complement; besides, any intersection of finitely many such subsets is either empty or has a finite complement. Thus, the properties (T1), (T2), (T3) are satisfies.

The finite complement topology is not metrisable if the set X is infinite,

EXAMPLE 7.8. The following example describes a variety of situations when different metrics define identical topologies. Suppose that

$$d,\, d' \,:\, X \times X \to \mathbf{R}$$

are two metrics on a set X and there exist constants $\alpha,\, \beta > 0$ such that for all $x, y \in X$ one has

$$\alpha \cdot d'(x, y) \leq d(x, y) \leq \beta \cdot d'(x, y). \tag{7.1}$$

Inequalities (2.6) is an example of inequalities (7.1). To show that topologies on X generated by d and d' coincide we need to show that (a) any open ball $B^d(x; r)$ is open with respect to the d'-topology and (b) any open ball $B^{d'}(x; r)$ is open with respect to the d-topology.

To show (a), consider a point $y \in B^d(x; r)$. Then $r_0 = d(x, y) < r$ and $B^d(y; r - r_0) \subset B^d(x; r)$. Using (7.1) we note that $d'(y, z) < \beta^{-1}(r - r_0)$ implies $d(y, z) < r - r_0$ and hence $d(x, z) < r$, i.e.

$$B^{d'}(y; \beta^{-1} \cdot (r - r_0)) \subset B^d(y; r - r_0) \subset B^d(x; r).$$

Our claim (b) follows similarly. Indeed, if $y \in B^{d'}(x; r)$ then $r_0 = d'(x, y) < r$ and we have

$$B^d(y; \alpha \cdot (r - r_0)) \subset B^{d'}(y; r - r_0) \subset B^{d'}(x; r).$$

These inclusions show that any d-open ball is a union d'-open balls and vice versa, any d'-open ball is a union of d-open balls.

Finally in this section we shall discuss the construction of *the subspace topology.* Let $(X, \mathcal{T}_X)$ be a topological space and let $Y \subset X$ be a subset. We can define a topology $\mathcal{T}_Y$ on Y as follows

$$\mathcal{T}_Y = \{U \cap Y; U \in \mathcal{T}_X\}.$$

In other words, an open subset of the subspace topology $(Y, \mathcal{T}_Y)$ is an intersection of an open subset of X with Y. We shall also say that the topology $\mathcal{T}_Y$ is *induced* by the topology $\mathcal{T}_X$ on Y.

As an example, consider the standard embedding $\mathbf{R}^m \subset \mathbf{R}^n$ where $m < n$. The topology on $\mathbf{R}^m$ induced by the standard topology of $\mathbf{R}^n$ coincides with the standard topology of $\mathbf{R}^m$. This means that any open subset of $\mathbf{R}^m$ can be represented as the intersection of an open subset of $\mathbf{R}^n$ with $\mathbf{R}^m$.

7.2. Closed Sets and the Closure

Let $(X, \mathcal{T})$ be a topological space. A subset $F \subset X$ is defined to be *closed* if its complement $X - F$ is open, i.e. $X - F \in \mathcal{T}$. Below we list the properties of closed sets:

(C1) The empty set $\emptyset$ and the whole space X are closed.
(C2) The intersection of any family of closed sets is closed.
(C3) The union of any finite collection of closed sets is a closed set.

The properties (C1), (C2), (C3) follow from (T1), (T2), (T3) by using de Morgan's law. For instance, we may prove (C2) as follows. Let F_α be a collection of closed subsets of X where $\alpha \in A$. The complement $X - F_\alpha$ is open and hence the union $\cup_\alpha(X - F_\alpha)$ is an open subset. Hence the complement $X - \cup_\alpha(X - F_\alpha)$ is closed. However, by de Morgan's law, this complement $X - \cup_\alpha(X - F_\alpha)$ equals the intersection $\cap_\alpha F_\alpha$, and therefore (C2) follows.

Definition 7.9. The closure $\overline{A}$ of a subset $A \subset X$ of a topological space $(X, \mathcal{T})$ is defined as the intersection of all closed subsets of X containing A, i.e.

$$\overline{A} = \bigcap_{F \supset A \text{ is closed}} F.$$

Clearly, $A \subset \overline{A}$ and the closure $\overline{A}$ is a closed subset as follows from (C2). In fact the closure $\overline{A}$ is the smallest closed subset containing A.

Theorem 7.10. *A point $x \in X$ belongs to the closure, $x \in \overline{A}$, if and only if every open subset $U \subset X$ containing x intersects A.*

Proof. The complement $X - \overline{A}$ is open since $\overline{A}$ is closed. Thus, $x \notin \overline{A}$ if and only if $x \in X - \overline{A}$, i.e. when there exists an open subset containing x and disjoint from A. □

The operation of closure has the following properties:

(1) $A \subset \overline{A}$;
(2) $\overline{\overline{A}} = \overline{A}$;
(3) $A_1 \subset A_2$ implies $\overline{A}_1 \subset \overline{A}_2$;
(4) $\overline{A_1 \cup A_2} = \overline{A}_1 \cup \overline{A}_2$.

Let us prove the last property. Since $\overline{A}_1 \cup \overline{A}_2$ is closed and contains $A_1 \cup A_2$ we obtain $\overline{A_1 \cup A_2} \subset \overline{A}_1 \cup \overline{A}_2$. On the other hand, $A_i \subset A_1 \cup A_2$, where $i = 1, 2$, imply $\overline{A}_i \subset \overline{A_1 \cup A_2}$ and hence $\overline{A}_1 \cup \overline{A}_2 \subset \overline{A_1 \cup A_2}$. □

As in the case of metric spaces, we may speak about limit points.

Definition 7.11. Let $A \subset X$ be a subset of a topological space $(X, \mathcal{T})$. A point $x \in X$ is a limit point (or accumulation or cluster point) of A if every open set containing x intersects A in some point other than x. In other words, x is a limit point of A if $x \in \overline{A - \{x\}}$.

We shall denote by A' the set of limit points of A.

Example 7.12. If $X = \mathbf{R}$ with the standard topology and $A = (0, 1]$ then the set of limit points is $A' = [0, 1]$. For $B = \{\frac{1}{n}; n = 1, 2, \dots\} \subset \mathbf{R}$ we have $B' = \{0\}$.

Theorem 7.13. *One has $\overline{A} = A \cup A'$.*

Proof. Clearly, $A' \subset \overline{A}$ and $A \subset \overline{A}$ implying $A \cup A' \subset \overline{A}$. On the other hand, if $x \in \overline{A}$ and $x \notin A$ then every open subset containing x intersects A and this intersection cannot not involve x, i.e. $x \in A'$. Therefore, $\overline{A} \subset A \cup A'$. □

Corollary 7.14. A subset $A \subset X$ is closed if and only if $A' \subset A$.

Example 7.15. Consider the finite complement topology of Example 7.7 on an infinite set X. Then any infinite subset $A \subset X$ is dense, i.e. its closure $\overline{A}$ equals X.