CHAPTER 6

Contraction Mappings and the Fixed Point Theorem

6.1. Fixed Point Theorem

Let (X, d) be a metric space.

DEFINITION 6.1. A map $f : X \to X$ is a contraction if there exists $\alpha < 1$ such that for all $x, y \in X$ one has

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y). \tag{6.1}$$

THEOREM 6.2 (Fixed Point Theorem). *Any contraction mapping $f : X \to X$ of a complete metric space X has a unique fixed point, i.e. a point $x \in X$ with $f(x) = x$.*

PROOF. Let $x_0 \in X$ be an arbitrary point. Define the sequence $x_n \in X$ by $x_1 = f(x_0)$, $x_2 = f(x_1)$, and in general $x_n = f(x_{n-1})$ where $n = 1, 2, \dots$. We claim that (x_n) is a Cauchy sequence. Indeed, for $n \leq m$ we have

$$\begin{aligned} d(x_n, x_m) &= d(f^n(x_0), f^n(x_{m-n})) \\ &\leq \alpha^n d(x_0, x_{m-n}) \\ &\leq \alpha^n [d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{m-n-1}, x_{m-n}) \\ &\leq \alpha^n [d(x_0, x_1) + \alpha d(x_0, x_1) + \alpha^2 d(x_0, x_1) + \dots \alpha^{m-n-1} d(x_0, x_1)] \\ &\leq d(x_0, x_1) \cdot \frac{\alpha^n}{1-\alpha} \end{aligned}$$

As $\alpha < 1$ we see that $d(x, x_m) \to 0$ as $n, m \to \infty$, i.e. the sequence (x_n) is a Cauchy sequence. Since the space X is complete the sequence x_n must have a limit which we denote $x_0 = \lim x_n$. Then $f(x_0) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = x_0$, i.e. x_0 is a fixed point of the map f.

The fixed point x_0 is unique: if y_0 is another fixed point, i.e. $f(y_0) = y_0$, then

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq \alpha \cdot d(x_0, y_0)$$

which can happen only if $d(x_0, y_0) = 0$, i.e. $x_0 = y_0$. □

EXAMPLE 6.3. Let $X = [1, \infty)$ with the usual metric; it is a complete metric space. Consider the following map $f : X \to X$,

$$f(x) = x + \frac{1}{x}, \quad x \in [1, \infty).$$

For $x, y \in X$ we have

$$\begin{aligned} f(x) - f(y) &= x - y + \frac{y - x}{xy} \\ &= (x - y)(1 - \frac{1}{xy}). \end{aligned}$$

Thus,

$$|f(x) - f(y)| = |x - y| \cdot |1 - \frac{1}{xy}| < |x - y|.$$

However f has no fixed points as the equation

$$f(x) = x + \frac{1}{x} = x$$

has no solutions. Figure 1 shows the graphs of the functions $f(x) = x + \frac{1}{x}$ and $y = x$ illustrating

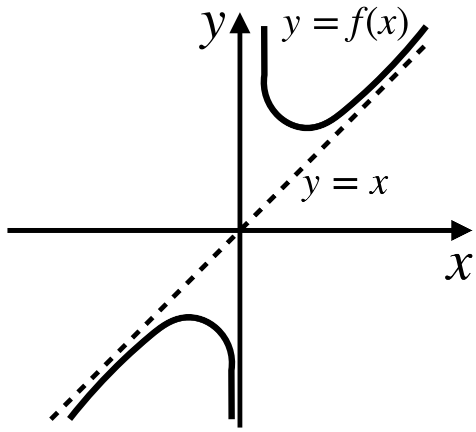

FIGURE 1. The graphs of the functions $y = x + \frac{1}{x}$ and $y = x$

the fact that these two curves have no intersection. This example shows that the inequality (6.1) cannot be replaced by a weaker inequality $d(f(x), f(y)) < d(x, y)$.

6.2. Applications of the Fixed Point Theorem

6.2.1. Let $f : [a, b] \to [a, b]$ be a C^1-smooth function satisfying $|f'(x)| \leq K < 1$ for all $x \in [a, b]$. Then f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq K \cdot |x - y|, \quad x, y \in [a, b]. \tag{6.2}$$

Indeed, by the Mean Value Theorem

$$f(y) - f(x) = f'(\xi) \cdot (y - x), \quad \text{for some} \quad \xi \in [x, y].$$

Inequality (6.2) is the contraction inequality (6.1) for the Euclidean metric on $\mathbf{R}$. Since the closed interval $[a, b]$ is complete, the Contraction Mapping Theorem is applicable and we obtain that the solution to the equation

$$f(x) = x$$

exists and is unique. Moreover, it can be found with arbitrarily small error by performing the iterations

$$x_n = f(x_{n-1}), \quad n = 1, 2, \ldots \tag{6.3}$$

where the initial approximation $x_0 \in [a, b]$ is chosen arbitrarily.

EXAMPLE 6.4. Consider the equation

$$\frac{1}{2} \cdot \cos x + 5 = x. \tag{6.4}$$

Denoting $f(x) = \frac{1}{2} \cdot \cos x + 5$ we have $f : \mathbf{R} \to \mathbf{R}$ and $f'(x) = -\frac{1}{2} \sin x$. Thus,

$$|f'(x)| \le \frac{1}{2},$$

i.e. $f : \mathbf{R} \to \mathbf{R}$ is a contraction mapping. Staring with $x_0 = 8$ we obtain

$x_1 = 4.927,$
$x_2 = 5.10,$
$x_3 = 5.19.$
$x_4 = 5.23.$
$x_5 = 5.24,$
$x_6 = 5.255,$
$x_7 = 5.258,$
$x_8 = 5.259,$
$x_9 = 5.26,$
$x_{10} = 5.26.$

6.2.2. Suppose that we need to find a solution of the equation $F(x) = 0$ lying in the interval $[a, b]$, where $F(a) < 0$ and $F(b) > 0$.

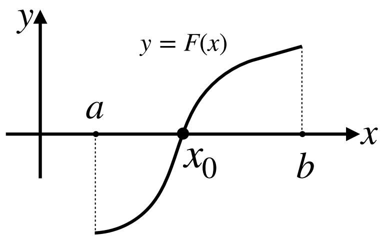

FIGURE 2. Equation $F(x) = 0$

Denote

$$f(x) = x - \lambda F(x), \quad \lambda > 0.$$

Then solutions of the equation $F(x) = 0$ are solutions of the fixed point equation $f(x) = x$ and vice versa. Suppose that

$$0 < K_1 \le F'(x) \le K_2 \quad \text{for} \quad x \in [a, b].$$

Then the derivative

$$f'(x) = 1 - \lambda F'(x)$$

satisfies

$$1 - \lambda K_2 \le f'(x) \le 1 - \lambda K_1.$$

We want

$$1 - \lambda K_1 < 1 \quad \text{and} \quad 1 - \lambda K_2 > -1.$$