MTH6127 **Metric Spaces and Topology**

Course work 6 **1 March 2024**

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1. Suppose that (X, d) is a metric space and $A \subseteq X$. Show that if $A \subseteq F \subseteq X$ where F is closed, then $\bar{A} \subseteq F$, where $\bar{A}$ is the closure of A.

2. Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine $\bar{A}$. Is A closed?

3. Let (X, d) be a metric space and $A \subset X$. Then the distance from x to A is defined as
$$\text{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$
Prove that $\overline{A} = \{x \in X; \text{dist}(x, A) = 0\}$.

4. The diameter of a metric space (X, d) is defined as $\sup\{d(x, y); x, y \in X\}$.
A set A in a metric space (X, d) is called bounded iff $\text{diam}(A) < \infty$. Prove that:
   (a) A is bounded if and only if there exist $x \in A$ and $r > 0$ such that $A \subset B(x; r)$,
   (b) Any finite set A is bounded,
   (c) A Cauchy sequence in (X, d) is a bounded set.

5. Assume that a Cauchy sequence (x_n) in a metric space (X, d) contains a subsequence (x_{n_i}) which converges to a point $x_0 \in X$. Show that the whole sequence (x_n) converges to x_0 as well.