5.4. Nested sequences of balls

THEOREM 5.9. *A metric space (X,d) is complete if and only if in X every nested sequence*

$$B_1 \supset B_2 \supset B_3 \supset \dots$$

of closed balls whose radii tend to zero has a non-empty intersection, $\cap_{n=1}^{\infty} B_n \neq \emptyset$.

PROOF. Suppose that the space X is complete. Given a nested sequence of closed balls B_n, where the radius r_n of B_n tends to zero, consider the sequence of their centres $x_n \in B_n$. For $m > n$ one has $d(x_n, x_m) \leq r_n$ and hence x_n is a Cauchy sequence. Let $x \in X$ denote the limit of the sequence x_n (which exists since we assume that X is complete). Since $x_m \in B_n$ for all $m \geq n$ we see that $x \in B_n$ for all n and hence $x \in \cap_{n=1}^{\infty} B_n$.

Conversely, suppose that in X the intersection of any nested sequence of closed balls $B_1 \supset B_2 \supset B_3 \supset \dots$ with radii tending to 0 is not empty. Let $x_n \in X$ be a Cauchy sequence. We may find an integer n_1 such that $d(x_n, x_{n_1}) < 1/2$ for all $n \geq n_1$. Similarly, we can find $n_2 > n_1$ such that $d(x_n, x_{n_2}) < 1/2^2$ for all $n \geq n_2$. Continuing by induction, for any integer k we can find $n_k > n_{k-1}$ such that for all $n \geq n_k$ one has $d(x_n, x_{n_k}) < 1/2^k$. Let B_k denote the closed ball $B[x_{n_k}; 1/2^{k-1}]$. Then $x_{n_{k+1}} \in B_k$ and moreover $B_{k+1} \subset B_k$ since $1/2^k + 1/2^{k+1} < 1/2^{k-1}$. By our assumption the intersection $\cap_{k=1}^{\infty} B_k$ contains a point x which then satisfies $d(x, x_{n_k}) \leq 1/2^{k-1}$ for all k. Hence, x is the limit of the subsequence x_{n_k} of the original Cauchy sequence. However, if a Cauchy sequence has a convergent subsequence then it converges as well. This completes the proof. □

5.5. Theorem of Baire

LEMMA 5.10. *For a subset $A \subset X$ of a metric space (X,d) the following two properties are equivalent:*

(a) the complement of the closure $X - \overline{A}$ is dense in X;

(b) every open ball $B \subset X$ contains another open ball $B' \subset B$ having no points of A, i.e. such that $B' \cap A = \emptyset$.

PROOF. Suppose that (a) is satisfied and let $B \subset X$ be an open ball. Then B must contain a point $x \notin \overline{A}$ and (since $\overline{A}$ is closed) an open ball $B' \subset B$ with centre x must lie in $X - \overline{A}$ implying that $B' \cap A = \emptyset$.

If $X - \overline{A}$ is not dense then there exists a non-empty open subset $U \subset X$ having no points of $X - \overline{A}$, i.e. $U \subset \overline{A}$ and (b) is not satisfied. □

DEFINITION 5.11. A subset $A \subset X$ of a metric space is nowhere dense if it satisfies the equivalent properties of Lemma 5.10

THEOREM 5.12 (Baire). *A complete metric space cannot be represented as the union of countably many nowhere dense subsets.*

PROOF. Suppose that $X = \cup_{n=1}^{\infty} M_n$ where each subset M_n is a nowhere dense subset of a complete metric space X. Let B_0 be an open ball of radius 1. Since M_1 is nowhere dense we may find a closed ball B_1 of radius less than 1/2 such that $B_1 \subset B_0$ and $B_1 \cap M_1 = \emptyset$. Similarly, the ball B_1 contains a closed ball B_2 of radius less than 1/3 having no points of M_2. We obtain a nested sequence of closed balls B_n with their radii tending to 0 and by Theorem 5.9 the intersection $\cap B_n$ is not empty, i.e. contains a point $x \in X$. Then $x \notin M_n$ for any n, contradiction. □

5.6. Completion of a metric space

Let (X, d) be a metric space which is not complete.

DEFINITION 5.13. A complete metric space (X^*, d^*) is a completion of (X, d) if X is isometric to a dense subset of X^*.

Equivalently, a complete metric space (X^*, d^*) is a completion of (X, d) if:

(1) $X \subset X^*$ and the metric on X is induced by the metric d^*,
(2) X is dense in X^*, i.e. $\overline{X} = X^*$.

EXAMPLE 5.14. $\mathbf{R}$ is a completion of $\mathbf{Q}$.

EXAMPLE 5.15. A completion of $X = \mathbf{R} - \{0\}$ is $\mathbf{R}$.

THEOREM 5.16. *Any metric space (X, d) admits a completion. The completion (X^*, d^*) is unique up to an isometry identical on X.*

LEMMA 5.17. *Suppose that $x_n \to x_0$ and $y_n \to y_0$ are two convergent sequences in a metric space (X, d). Then the numerical sequence $d(x_n, y_n) \in \mathbf{R}$ converges to $d(x_0, y_0)$.*

PROOF. We have the inequalities $d(x_n, y_n) \le d(x_n, x_0) + d(x_0, y_0) + d(y_0, y_n)$ and $d(x_0, y_0) \le d(x_0, x_n) + d(x_n, y_n) + d(y_n, y_0)$ which give $d(x_n, y_n) - d(x_0, y_0) \le d(x_n, x_0) + d(y_0, y_n)$ and $d(x_0, y_0) - d(x_n, y_n) \le d(x_0, x_n) + d(y_n, y_0)$, i.e,

$$|d(x_n, y_n) - d(x_0, y_0)| \le d(x_n, x_0) + d(y_0, y_n). \tag{5.11}$$

The RHS tends to 0, hence $d(x_n, y_n) \to d(x_0, y_0)$. □

PROOF OF THEOREM 5.16. We shall first prove *the uniqueness*. Suppose that $X \subset X^*$ and $X \subset Y^*$ are two completions. We want to construct an isometry $f : X^* \to Y^*$ which is identical on X, i.e. $f(x) = x$ for $x \in X$. To define $f(x^*)$ where $x^* \in X^*$ we use the fact that X is dense in X^* and find a sequence $x_n \in X$ with $x_n \to x^*$. The sequence $x_n \in X \subset Y^*$ is Cauchy and has a limit $y^* = \lim x_n$ where $y^* \in Y^*$.

We shall show that (a) y^* depends only on x^*; (b) the map $f : X^* \to Y^*$ given by $f(x^*) = y^*$ is an isometry; (c) $f(x) = x$ for every $x \in X$.

To prove (a) consider another sequence $x'_n \in X$ converging to x^* in X^*. If $z^* \in Y^*$ is its limit in Y^* then using Lemma 5.17 we have

$$d_{Y^*}(y^*, z^*) = \lim d_{Y^*}(x_n, x'_n) = \lim d_{X^*}(x_n, x'_n) = d_{X^*}(x^*, x^*) = 0$$

i.e. $y^* = z^*$.

Next we prove that $f : X^* \to Y^*$ is an isometry. For x_1^*, $x_2^* \in X^*$ find two sequences $x_n \to x_1^*$ and $x'_n \to x_2^*$ in X^*. Then

$$d_{X^*}(x_1^*, x_2^*) = \lim d_{X^*}(x_n, x'_n) = \lim d_{Y^*}(x_n, x'_n) = d_{Y^*}(f(x_1^*), f(x_2^*)).$$

Finally, to prove (c) we note that for $x \in X$ we may take $x_n = x$ (the stationary sequence) and hence the above procedure applied to this sequence gives $f(x) = x$.

Now we shall prove *the existence* of a completion X^*. Consider the set of all Cauchy sequences (x_n) in a given metric space (X, d) and introduce in this set the following equivalence relation: $(x_n) \sim (y_n)$ iff $d(x_n, y_n) \to 0$. Define X^* as the set of equivalence classes.

The inclusion $X \subset X^*$ is given by associating with $x \in X$ the class of the stationary sequence $x_n = x$.

To define *a metric on* X^* we note that for two Cauchy sequences (x_n) and (y_n) in X the limit $\lim d(x_n, y_n)$ exists. Indeed,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, y_n) + d(x_m, y_m) \tag{5.12}$$

and the sequence of real numbers $d(x_n, y_n)$ is a Cauchy sequence. To show (5.12) we note

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and similarly

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m).$$

These two inequalities give (5.12). Thus we may define a metric on X^* by the formula

$$d_{X^*}((x_n), (y_n)) = \lim d(x_n, y_n). \tag{5.13}$$

To show (M1) we note that $d_{X^*}((x_n), (y_n)) \geq 0$ and if $d_{X^*}((x_n), (y_n))$ then $\lim d(x_n, y_n) = 0$, i.e. the Cauchy sequences (x_n) and (y_n) are equivalent. (M2) and (M3) are obvious, they follow from the corresponding properties of the metric d.

Note that X *is dense* in X^*. Indeed, given a point $x^* \in X^*$ represented by a Cauchy sequence (x_n) in X, we see that

$$\lim_{n\to\infty} d_{X^*}(x_n, x^*) = \lim_{n\to\infty}[\lim_{m\to\infty} d(x_n, x_m)] = \lim_{n,m\to\infty} d(x_n, x_m) = 0$$

which means that $x_n \to x^*$ in X^*.

Finally we show that the metric space X^* *is complete.* Consider a Cauchy sequence (x_n^*) in X^*. Since X is dense in X^* we may find $x_n \in X$ with $d_{X^*}(x_n, x_n^*) < 1/n$. We see that

$$d(x_n, x_m) \leq d_{X^*}(x_n, x_n^*) + d_{X^*}(x_n^*, x_m^*) + d_{X^*}(x_m^*, x_m) < d_{X^*}(x_n^*, x_m^*) + 1/n + 1/m$$

implying that (x_n) is a Cauchy sequence. Let $x_0^* \in X^*$ be the equivalence class of this sequence. Then

$$d_{X^*}(x_n^*, x_0^*) \leq d_{X^*}(x_n, x_0^*) + 1/n = \lim_{m\to\infty} d(x_n, x_m) + 1/n$$

and we see that $d_{X^*}(x_n^*, x_0^*)$ tends to zero which means that the sequence (x_n^*) converges to x_0^* in X^*.

This completes the proof. □

CHAPTER 6

Contraction mappings and the fixed point theorem

6.1. Contraction mappings

Let (X, d) be a metric space.

DEFINITION 6.1. A map $f : X \to X$ is a contraction if there exists $\alpha < 1$ such that for all $x, y \in X$ one has

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y). \tag{6.1}$$

THEOREM 6.2 (Fixed Point Theorem). *Any contraction mapping $f : X \to X$ of a complete metric space X has a unique fixed point, i.e. a point $x \in X$ with $f(x) = x$.*

PROOF. Let $x_0 \in X$ be an arbitrary point. Define the sequence $x_n \in X$ by $x_1 = f(x_0)$, $x_2 = f(x_1)$, and in general $x_n = f(x_{n-1})$ where $n = 1, 2, \ldots$. We claim that (x_n) is a Cauchy sequence. Indeed, for $n \leq m$ we have

$$\begin{aligned} d(x_n, x_m) &= d(f^n(x_0), f^n(x_{m-n})) \\ &\leq \alpha^n d(x_0, x_{m-n}) \\ &\leq \alpha^n [d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-n-1}, x_{m-n}) \\ &\leq \alpha^n [d(x_0, x_1) + \alpha d(x_0, x_1) + \alpha^2 d(x_0, x_1) + \ldots \alpha^{m-n-1} d(x_0, x_1)] \\ &\leq d(x_0, x_1) \cdot \frac{\alpha^n}{1-\alpha} \end{aligned}$$

As $\alpha < 1$ we see that $d(x, x_m) \to 0$ as $n, m \to \infty$, i.e. the sequence (x_n) is a Cauchy sequence. Since the space X is complete the sequence x_n must have a limit which we denote $x_0 = \lim x_n$. Then $f(x_0) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = x_0$, i.e. x_0 is a fixed point of the map f.

The fixed point x_0 is unique: if y_0 is another fixed point, i.e. $f(y_0) = y_0$, then

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq \alpha \cdot d(x_0, y_0)$$

which can happen only if $d(x_0, y_0) = 0$, i.e. $x_0 = y_0$. □

EXAMPLE 6.3. Let $X = [1, \infty)$ with the usual metric; it is a complete metric space. Consider the following map $f : X \to X$

$$f(x) = x + \frac{1}{x}, \quad x \in [1, \infty).$$

For $x, y \in X$ we have

$$\begin{aligned} f(x) - f(y) &= x - y + \frac{y - x}{xy} \\ &= (x - y)(1 - \frac{1}{xy}). \end{aligned}$$