CHAPTER 5

Complete Metric Spaces

5.1. First examples of complete metric spaces

Definition 5.1. In a metric space (X, d), a sequence of points (x_n) is a Cauchy sequence if for any $\epsilon > 0$ there is $N > 0$ such that for all $n, m > N$ one has $d(x_n, x_m) < \epsilon$.

Lemma 5.2. *Any convergent sequence is a Cauchy sequence.*

Proof. If $x_n \to x_0$ then for any $\epsilon > 0$ there is $N > 0$ such that for all $n > N$ one has $d(x_n, x_0) < \epsilon/2$. Then for $n, m > N$ one has $d(x_n, x_n) \leq d(x_n, x_0) + d(x_m, x_0) < \epsilon$. □

Definition 5.3. A metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Example 5.4. $X = \mathbf{R}$ with the usual metric is complete. This is one of the axioms defining the set of real numbers.

The set $Y = \mathbf{R} - \{0\}$ with the induced metric is not complete as the sequence $x_n = 1/n$ is Cauchy but has no limit in Y.

Example 5.5. The metric space $\mathbf{R}^m$ with d_∞ metric s complete. Indeed, consider a Cauchy sequence $v_n = (x_1^n, x_2^n, \ldots, x_m^n) \in \mathbf{R}^m$. This means that for any $\epsilon > 0$ there is $N > 0$ such that

$$\max\{|x_i^n - x_i^m|;\ i = 1, 2, \ldots m\} < \epsilon,$$

i.e. $|x_i^n - x_i^m| < \epsilon$ for every $i = 1, 2, \ldots, m$. We see that each coordinate sequence $x_i^1, x_i^2, x_i^3, \ldots, x_i^n, \ldots$ is a Cauchy sequence of real numbers. Since $\mathbf{R}$ is complete, we obtain that $x_i^n \to x_i^0$ and therefore the sequence v_n converges to the vector $v_0 = (x_1^0, x_2^0, \ldots, x_m^0)$ in $(\mathbf{R}^m, d_\infty)$.

Example 5.6. The metric space $\mathbf{R}^m$ with d_p metric s complete for any $p \in [1, \infty]$. The case $p = \infty$ has been discussed above and for a general $p \in [1, \infty)$ we can use the inequalities

$$d_\infty(v, w) \leq d_p(v, w) \leq m^{\frac{1}{p}} \cdot d_\infty(v, w), \quad v, w \in \mathbf{R}^m,$$

see (2.6). It follows that a sequence of vectors $v_n \in \mathbf{R}^m$ is a Cauchy sequence with respect to the metric d_∞ if and only if it is a Cauchy sequence with respect to d_p. Similarly, a sequence $v_n \in \mathbf{R}^m$ converges to $v_0 \in \mathbf{R}^m$ with respect to d_∞ if and only if it converges to v_0 with respect to d_p.

Proposition 5.7. *Let (X, d) be a complete metric space. A subset $Y \subset X$ viewed with the induced metric is complete if and only if it is closed.*

Proof. Any Cauchy sequence in Y is a Cauchy sequence in X and has a limit in X since X is complete. If Y is closed the limit point must belong to Y, see Lemma 4.21.

If the subset $Y \subset X$ is not closed we may find a sequence of points $x_n \in Y$ having its limit $x_0 \in X - Y$ (by Lemma 4.21). This sequence is Cauchy in Y and has no limit in Y. Hence Y is not complete. □

5.2. Banach and Hilbert spaces

DEFINITION 5.8. A normed space $(V, ||\cdot||)$ is called *a Banach space* if the associated metric $d(v,w) = ||v-w||$ is complete.

A scalar product space $(V, \langle\, ,\, \rangle)$ is called *a Hilbert space* of the associated normed space $(V, ||\cdot||)$, where $||v||^2 = \langle v, v\rangle$ is a Banach space.

As our first example consider the space $V = C[a,b]$ of continuous functions on $[a,b]$ with the metric

$$d(f,g) = ||f-g|| = \max_{t\in[a,b]} |f(t)-g(t)|.$$

Let us show that this space is complete and hence is a Banach space. Let $f_n(t) \in C[a,b]$ be a Cauchy sequence. Then for any $\epsilon > 0$ there is $N > 0$ such that for all $n, m > N$ one has

$$|f_n(t) - f_m(t)| < \epsilon \quad \text{for any} \quad t \in [a,b]. \tag{5.1}$$

For a fixed $t \in [a,b]$ the sequence $f_n(t)$ is a Cauchy sequence of real numbers and hence it has a limit which we shall denote $f(t)$. Thus, for every $t \in [a,b]$, $\lim f_n(t) = f(t)$ pointwise. Taking in (5.1) the limit with respect to $m \to \infty$ we obtain

$$|f_n(t) - f(t)| < \epsilon, \quad \text{for any} \quad t \in [a,b]. \tag{5.2}$$

Thus we see that the sequence of continuous functions $f_n(t)$ converges to $f(t)$ *uniformly.* We know from the calculus courses that the limit function $f(t)$ must be continuous, $f(t) \in C[a,b]$, and the inequality (5.2) shows that the sequence f_n converges to f in $C[a,b]$.

Consider now a different norm on $C[a,b]$ which is called the L^2-norm:

$$||f|| = \left[\int_a^b f(t)^2 dt\right]^{1/2}, \quad f \in C[a,b]. \tag{5.3}$$

It is an inner product space where the scalar product is given by

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt, \quad f, g \in C[a,b]. \tag{5.4}$$

and the conditions (N1), (N2) and (N3) are satisfied. Let us show that the metric (5.3) is not complete. For simplicity we shall assume that $[a,b] = [-1,1]$ but clearly the same result hold for any closed interval $[a,b]$.

For $t \in [-1,1]$ define $f_n(t)$ by

$$f_n(t) = \begin{cases} 1, & \text{if} \quad t \in [1/n, 1], \\ -1, & \text{if} \quad t \in [-1, -1/n], \\ nt, & \text{if} \quad t \in [-1/n, 1/n]. \end{cases}$$

Clearly, f_n is continuous, $f_n \in C[-1,1]$. Let us show that the sequence f_n is a Cauchy sequence with respect to the metric associated with the norm (5.3). For $n, m > N$ one has

$$|f_n(t) - f_m(t)| = \begin{cases} 0, & \text{if} \quad t \notin [-1/N, 1/N], \\ \leq 2, & \text{for} \quad t \in [-1/N, 1/N]. \end{cases}$$

Therefore,

$$\int_{-1}^{1} (f_n(t) - f_m(t))^2\, dt \le 2^2 \cdot \frac{2}{N} = \frac{8}{N}.$$

Given an arbitrary $\epsilon > 0$ we can take $N = \frac{8}{\epsilon^2}$. Then for $n, m > N$ one has

$$\int_{-1}^{1} (f_n(t) - f_m(t))^2\, dt \le \epsilon^2, \quad \text{i.e.} \quad ||f_n - f_m|| \le \epsilon,$$

which shows that the sequence f_n is a Cauchy sequence. The graph of f_n and the graph of the point-

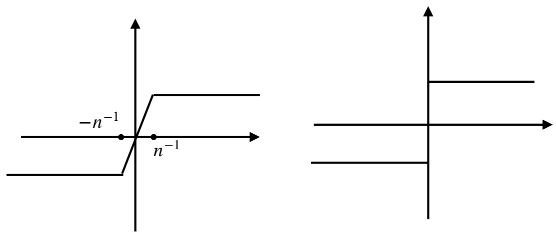

FIGURE 1. The function $f_n(t)$ (left) and its pointwise limit $f(t)$ (right)

wise limit $f(t) = \lim f_n(t)$ are shown on Figure 1. The function $f(t)$ is discontinuous, $f \notin C[-1, 1]$, and we see that the sequence f_n has no limit in the space $C[-1, 1]$ with respect to the L^2-metric.

Indeed, let us assume that the sequence f_n converges to a continuous function $\psi \in C[-1, 1]$ with respect to the metric determined by the norm (5.3). Suppose that for some $t_0 \in (0, 1)$ one has $\psi(t_0) \ne 1$. Using the continuity of ψ we conclude that there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies $|\psi(t) - 1| > 1/2 \cdot |\psi(t_0) - 1| = a > 0$. Hence for sufficiently large n we shall have $||\psi - f_n|| \ge a\delta$ contradicting the assumption that $f_n \to \psi$. This shows that the value $\psi(t_0)$ must be equal 1 for all $t_0 \in (0, 1]$. Similarly, one shows that ψ must be equal to -1 on the interval $[-1, 0)$. Clearly, no continuous function on $[-1, 1]$ with these two properties exists.

5.3. The space ℓ^2

The elements of the space ℓ^2 are infinite sequences $v = (x_1, x_2, \dots)$ of real numbers satisfying the condition

$$||v||^2 = \sum_{i=1}^{\infty} x_i^2 < \infty. \tag{5.5}$$

If $w = (y_1, y_2, \dots)$ is another infinite sequence satisfying (5.5) then their sum

$$v + w = (x_1 + y_1, x_2 + y_2, \dots)$$

also satisfies (5.5). Indeed, taking a large integer $M > 0$, denote by v^M and w^M the M-dimensional vectors having as coordinates the first M coordinates of v and w correspondingly. By the Minkowski inequality (1.19),

$$||v^M + w^M|| \le ||v^M|| + ||w^M|| \le ||v|| + ||w|| \tag{5.6}$$

which shows that the monotone function $M \mapsto ||v^M + w^M||$ is bounded and hence has a limit when $M \to \infty$, i.e. $||v + w|| < \infty$. Thus, ℓ^2 is a vector space.

For $v, w \in \ell^2$ their scalar product is defined by

$$\langle v, w\rangle = \sum_{i=1}^{\infty} x_i y_i, \tag{5.7}$$

where $v_i = (x_1, x_2, \dots)$ and $w = (y_1, y_2, \dots)$.

To show that (5.7) is well-defined, i.e. it is finite, we write the Cauchy inequality (3.4) for the finite dimensional vectors v^M and w^M:

$$\langle v^M, w^M\rangle \;=\; \sum_{i=1}^{M} x_i y_i \;\le\; \sum_{i=1}^{M} |x_i|\cdot|y_i| \;\le\; ||v^M||\cdot||w^M|| \;\le\; ||v||\cdot||w||. \tag{5.8}$$

This shows that the series (5.7) converges absolutely.

Next we show that the inner product space ℓ^2 is complete with respect to its metric, i.e. it is a Hilbert space. Consider a Cauchy sequence $v_n \in \ell^2$. Then for any $\epsilon > 0$ one has

$$||v_n - v_m|| < \epsilon \tag{5.9}$$

assuming that $n, m > N = N(\epsilon)$. The inequality (5.9) means that

$$\sum_{i=1}^{\infty} |x_i(n) - x_i(m)|^2 < \epsilon^2.$$

Therefore, each coordinate sequence $x_i(1), x_i(2), \dots$ is a Cauchy sequence of reals, thus it converges to a limit which we shall denote by $x_i \in \mathbf{R}$.

Denote $u = (x_1, x_2, \dots)$. We shall show below that $u \in \ell^2$ and the sequence v_n converges to u in ℓ^2. We may write

$$\sum_{i=1}^{\infty} |x_i(n) - x_i(m)|^2 \;=\; \sum_{i=1}^{M} |x_i(n) - x_i(m)|^2 + \sum_{i=M+1}^{\infty} |x_i(n) - x_i(m)|^2 < \epsilon^2.$$

and hence $\sum_{i=1}^{M} |x_i(n) - x_i(m)|^2 < \epsilon$. Fix m and pass to the limit with respect to n. We obtain

$$\sum_{i=1}^{M} |x_i - x_i(m)|^2 < \epsilon$$

and since this is true for any M we obtain that the sum

$$\sum_{i=1}^{\infty} |x_i - x_i(m)|^2 \;\to\; 0 \tag{5.10}$$

tends to 0 when $m \to \infty$. We have

$$\sum_{i=1}^{\infty} |x_i|^2 \le \sum_{i=1}^{\infty} |x_i - x_i(m)|^2 + \sum_{i=1}^{\infty} |x_i(m)|^2.$$

This shows that $\sum_{i=1}^{\infty} |x_i|^2$ is finite, i.e. $u \in \ell^2$. Then (5.10) means that $||v_m - u|| \to 0$, i.e. v_n converges to u.

5.4. Nested sequences of balls

THEOREM 5.9. *A metric space (X, d) is complete if and only if in X every nested sequence*

$$B_1 \supset B_2 \supset B_3 \supset \dots$$

of closed balls whose radii tend to zero has a non-empty intersection, $\cap_{n=1}^{\infty} B_n \neq \emptyset$.

PROOF. Suppose that the space X is complete. Given a nested sequence of closed balls B_n, where the radius r_n of B_n tends to zero, consider the sequence of their centres $x_n \in B_n$. For $m > n$ one has $d(x_n, x_m) \leq r_n$ and hence x_n is a Cauchy sequence. Let $x_0 \in X$ denote the limit of the sequence x_n (which exists since we assume that X is complete). As $x_m \in B_n$ for all $m \geq n$, we see that $x_0 \in B_n$ for all n and hence $x_0 \in \cap_{n=1}^{\infty} B_n$, i.e. the intersection of the sequence of balls in non-empty.

Conversely, suppose that in the metric space X the intersection of any nested sequence of closed balls $B_1 \supset B_2 \supset B_3 \supset \dots$ with radii tending to 0 is not empty. Let $x_n \in X$ be a Cauchy sequence. We may find an integer n_1 such that $d(x_n, x_{n_1}) < 1/2$ for all $n \geq n_1$. Similarly, we can find $n_2 > n_1$ such that $d(x_n, x_{n_2}) < 1/2^2$ for all $n \geq n_2$. Continuing by induction, for any integer k we can find $n_k > n_{k-1}$ such that for all $n \geq n_k$ one has $d(x_n, x_{n_k}) < 1/2^k$. Let B_k denote the closed ball $B[x_{n_k}; 1/2^{k-1}]$. Then $x_{n_{k+1}} \in B_k$ and moreover

$$B_{k+1} \subset B_k$$

since for $x \in B_{k+1}$ one has

$$d(x, x_{n_k}) \leq d(x, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \leq 1/2^k + 1/2^k = 1/2^{k-1}.$$

By our assumption the intersection $\cap_{k=1}^{\infty} B_k$ contains a point x which then satisfies $d(x, x_{n_k}) \leq 1/2^{k-1}$ for all k. Hence, x is the limit of the subsequence x_{n_k} of the original Cauchy sequence. However, if a Cauchy sequence has a convergent subsequence then it converges as well. This completes the proof. □

5.5. Theorem of Baire

LEMMA 5.10. *For a subset $A \subset X$ of a metric space (X, d) the following two properties are equivalent:*

(a) the complement of the closure $X - \overline{A}$ is dense in X;

(b) every open ball $B \subset X$ contains another open ball $B' \subset B$ having no points of A, i.e. such that $B' \cap A = \emptyset$.

PROOF. Suppose that (a) is satisfied and let $B \subset X$ be an open ball. Then B must contain a point $x \notin \overline{A}$ and (since $\overline{A}$ is closed) an open ball $B' \subset B$ with centre x must lie in $X - \overline{A}$ implying that $B' \cap A = \emptyset$.

If $X - \overline{A}$ is not dense then there exists a non-empty open subset $U \subset X$ having no points of $X - \overline{A}$, i.e. $U \subset \overline{A}$ and (b) is not satisfied. □

DEFINITION 5.11. A subset $A \subset X$ of a metric space is nowhere dense if it satisfies the equivalent properties of Lemma 5.10

THEOREM 5.12 (Baire). *A complete metric space cannot be represented as the union of countably many nowhere dense subsets.*

PROOF. Suppose that $X = \cup_{n=1}^{\infty} M_n$ where each subset M_n is a nowhere dense subset of a complete metric space X. Let B_0 be an open ball of radius 1. Since M_1 is nowhere dense we may find a closed ball B_1 of radius less than $1/2$ such that $B_1 \subset B_0$ and $B_1 \cap M_1 = \emptyset$. Similarly, the ball B_1 contains a closed ball B_2 of radius less than $1/3$ having no points of M_2. We obtain a nested sequence of closed balls B_n with their radii tending to 0 and by Theorem 5.9 the intersection $\cap B_n$ is not empty, i.e. contains a point $x \in X$. Then $x \notin M_n$ for any n, contradiction. □

5.6. Completion of a metric space

Let (X, d) be a metric space which is not complete.

DEFINITION 5.13. A complete metric space (X^*, d^*) is a completion of (X, d) if X is isometric to a dense subset of X^*.

Equivalently, a complete metric space (X^*, d^*) is a completion of (X, d) if:

(1) $X \subset X^*$ and the metric on X is induced by the metric d^*,
(2) X is dense in X^*, i.e. $\overline{X} = X^*$.

EXAMPLE 5.14. $\mathbf{R}$ is a completion of $\mathbf{Q}$.

EXAMPLE 5.15. A completion of $X = \mathbf{R} - \{0\}$ is $\mathbf{R}$.

THEOREM 5.16. *Any metric space (X, d) admits a completion. The completion (X^*, d^*) is unique up to an isometry identical on X.*

LEMMA 5.17. *Suppose that $x_n \to x_0$ and $y_n \to y_0$ are two convergent sequences in a metric space (X, d). Then the numerical sequence $d(x_n, y_n) \in \mathbf{R}$ converges to $d(x_0, y_0)$.*

PROOF. We have the inequalities $d(x_n, y_n) \le d(x_n, x_0) + d(x_0, y_0) + d(y_0, y_n)$ and $d(x_0, y_0) \le d(x_0, x_n) + d(x_n, y_n) + d(y_n, y_0)$ which give $d(x_n, y_n) - d(x_0, y_0) \le d(x_n, x_0) + d(y_0, y_n)$ and $d(x_0, y_0) - d(x_n, y_n) \le d(x_0, x_n) + d(y_n, y_0)$, i.e,

$$|d(x_n, y_n) - d(x_0, y_0)| \le d(x_n, x_0) + d(y_0, y_n). \tag{5.11}$$

The RHS tends to 0, hence $d(x_n, y_n) \to d(x_0, y_0)$. □

PROOF OF THEOREM 5.16. We shall first prove *the uniqueness*. Suppose that $X \subset X^*$ and $X \subset Y^*$ are two completions. We want to construct an isometry $f : X^* \to Y^*$ which is identical on X, i.e. $f(x) = x$ for $x \in X$. To define $f(x^*)$ where $x^* \in X^*$ we use the fact that X is dense in X^* and find a sequence $x_n \in X$ with $x_n \to x^*$. The sequence $x_n \in X \subset Y^*$ is Cauchy and has a limit $y^* = \lim x_n$ where $y^* \in Y^*$.

We shall show that (a) y^* depends only on x^*; (b) the map $f : X^* \to Y^*$ given by $f(x^*) = y^*$ is an isometry; (c) $f(x) = x$ for every $x \in X$.

To prove (a) consider another sequence $x'_n \in X$ converging to x^* in X^*. If $z^* \in Y^*$ is its limit in Y^* then using Lemma 5.17 we have

$$d_{Y^*}(y^*, z^*) = \lim d_{Y^*}(x_n, x'_n) = \lim d_{X^*}(x_n, x'_n) = d_{X^*}(x^*, x^*) = 0$$

i.e. $y^* = z^*$.

Next we prove that $f : X^* \to Y^*$ is an isometry. For x_1^*, $x_2^* \in X^*$ find two sequences $x_n \to x_1^*$ and $x'_n \to x_2^*$ in X^*. Then

$$d_{X^*}(x_1^*, x_2^*) = \lim d_{X^*}(x_n, x'_n) = \lim d_{Y^*}(x_n, x'_n) = d_{Y^*}(f(x_1^*), f(x_2^*)).$$

Finally, to prove (c) we note that for $x \in X$ we may take $x_n = x$ (the stationary sequence) and hence the above procedure applied to this sequence gives $f(x) = x$.

Now we shall prove *the existence* of a completion X^*. Consider the set of all Cauchy sequences (x_n) in a given metric space (X, d) and introduce in this set the following equivalence relation: $(x_n) \sim (y_n)$ iff $d(x_n, y_n) \to 0$. Define X^* as the set of equivalence classes.

The inclusion $X \subset X^*$ is given by associating with $x \in X$ the class of the stationary sequence $x_n = x$.

To define *a metric on* X^* we note that for two Cauchy sequences (x_n) and (y_n) in X the limit $\lim d(x_n, y_n)$ exists. Indeed,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, y_n) + d(x_m, y_m) \tag{5.12}$$

and the sequence of real numbers $d(x_n, y_n)$ is a Cauchy sequence. To show (5.12) we note

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and similarly

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m).$$

These two inequalities give (5.12). Thus we may define a metric on X^* by the formula

$$d_{X^*}((x_n), (y_n)) = \lim d(x_n, y_n). \tag{5.13}$$

To show (M1) we note that $d_{X^*}((x_n), (y_n)) \geq 0$ and if $d_{X^*}((x_n), (y_n))$ then $\lim d(x_n, y_n) = 0$, i.e. the Cauchy sequences (x_n) and (y_n) are equivalent. (M2) and (M3) are obvious, they follow from the corresponding properties of the metric d.

Note that X *is dense* in X^*. Indeed, given a point $x^* \in X^*$ represented by a Cauchy sequence (x_n) in X, we see that

$$\lim_{n\to\infty} d_{X^*}(x_n, x^*) = \lim_{n\to\infty} [\lim_{m\to\infty} d(x_n, x_m)] = \lim_{n,m\to\infty} d(x_n, x_m) = 0$$

which means that $x_n \to x^*$ in X^*.

Finally we show that the metric space X^* *is complete.* Consider a Cauchy sequence (x_n^*) in X^*. Since X is dense in X^* we may find $x_n \in X$ with $d_{X^*}(x_n, x_n^*) < 1/n$. We see that

$$d(x_n, x_m) \leq d_{X^*}(x_n, x_n^*) + d_{X^*}(x_n^*, x_m^*) + d_{X^*}(x_m^*, x_m) < d_{X^*}(x_n^*, x_m^*) + 1/n + 1/m$$

implying that (x_n) is a Cauchy sequence. Let $x_0^* \in X^*$ be the equivalence class of this sequence. Then

$$d_{X^*}(x_n^*, x_0^*) \leq d_{X^*}(x_n, x_0^*) + 1/n = \lim_{m\to\infty} d(x_n, x_m) + 1/n$$

and we see that $d_{X^*}(x_n^*, x_0^*)$ tends to zero which means that the sequence (x_n^*) converges to x_0^* in X^*.

This completes the proof. □