MTH6127 **Metric Spaces and Topology**

Course work 4 **16 February 2024**

1. In this question the set of real numbers $\mathbb{R}$ is equipped with the Euclidean metric. Which of the following sets are open or closed? Explain your answer.

   (a) $(0, 1)$,

   (b) $(0, \infty)$,

   (c) $[0, \infty)$,

   (d) $\mathbb{R} - \mathbb{Z}$,

   (e) $\mathbb{Q}$.

2. Let $f_0 : [0, 1] \to \mathbb{R}$ be any continuous function on $[0, 1]$. Denote the identity function by I, so that $I(x) = x$ for all x. Define the sequence (f_n) of functions in $C[0, 1]$ by

$$f_n(x) = \frac{1}{2}(f_{n-1}(x) + I(x)),$$

   for all $n \geq 1$. Prove that

$$d(f_n, I) = \frac{1}{2} d(f_{n-1}, I)$$

   where d is the sup-metric on $C[0, 1]$. Deduce that the sequence (f_n) converges in $(C[0, 1], d)$ to I.

3. Consider the following metric d^* on $\mathbb{R}^2$:

$$d^*(p, q) = d^*\big((p_1, p_2), (q_1, q_2)\big) = \begin{cases} |p_1 - q_1|, & \text{if } p_2 = q_2; \\ |p_1 - q_1| + 1, & \text{otherwise.} \end{cases}$$

   Which of the sets below are open in the metric space $(\mathbb{R}^2, d^*)$? Justify your answers.

   (a) $[-1, 1] \times [-1, 1]$,

   (b) $(-1, 1) \times [-1, 1]$,

   (c) $\{p : \|p\| = p_1^2 + p_2^2 < 1\}$,

   (d) $\{p : \|p\| = p_1^2 + p_2^2 \leq 1\}$.

   Hint. How does an open ball with radius $0 < r < 1$ look like in $(\mathbb{R}^2, d^*)$?