## 4.6. Dense subsets

DEFINITION 4.25. In a metric space $(X, d)$ a subset $A \subset X$ is dense if $\overline{A} = X$.

EXAMPLE 4.26. The set of rationals $\mathbb{Q}$ is dense in $X = \mathbf{R}$ as any real number is a limit of a sequence of rational numbers. Similarly, $\mathbb{Q}^m \subset \mathbf{R}^m$ is dense with respect to any of the metrics $d_p$ where $p \in [1, \infty]$.

LEMMA 4.27. *A subset $A \subset X$ is dense if and only if every non-empty open subset of $X$ contains a point of $A$.*

PROOF. Suppose that $A \subset X$ is dense, i.e. $\overline{A} = X$, and $U \subset X$ is a non-empty open subset with $A \cap U = \emptyset$. Then $A$ is contained in the closed set $(X - U)$ and hence $\overline{X} \subset (X - U)$ - contradiction.

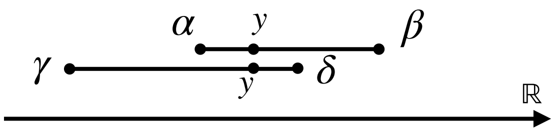
Now, let us assume that a subset $A \subset X$ is such that every non-empty open subset of $X$ contains a point of $A$. Then $\overline{A} = X$ since otherwise the set $U = X - \overline{A}$ would be non-empty, open and disjoint from $A$. □

## 4.7. Open subsets of R

Here we consider the real line $\mathbf{R}$ with the standard metric.

THEOREM 4.28. *Any open subset $U \subset \mathbf{R}$ is the union of a finite or countable collection of pairwise disjoint open intervals.*

PROOF. Let $U \subset \mathbf{R}$ be open. For $x, y \in U$ we shall write $x \sim y$ if there exists an open interval $(\alpha, \beta)$ such that $x, y \in (\alpha, \beta) \subset U$. We note that $\sim$ is an equivalence relation: if $x \sim y$ and $y \sim z$ then $x, y \in (\alpha, \beta) \subset U$ and $y, z \in (\gamma, \delta) \subset U$ then $(\alpha, \beta) \cup (\gamma, \delta) = (\alpha', \delta')$ where $\alpha' = \min\{\alpha, \gamma\}$ and $\delta' = \max\{\beta, \delta\}$. Thus, $x, z \in (\alpha', \delta') \subset U$.




Consider the partition of $U$ into the equivalence classes with respect to $\sim$

$$U = \bigsqcup_{\tau} I_\tau. \tag{4.2}$$

We claim that each equivalence class $I_\tau$ is an open interval $(\alpha_\tau, \beta_\tau)$, where $\alpha_\tau = \inf I_\tau$ and $\beta_\tau = \sup I_\tau$. The inclusion $I_\tau \subset (\alpha_\tau, \beta_\tau)$ is obvious since clearly $\alpha_\tau, \beta_\tau \notin I_\tau$. If $x, y \in I_\tau$ then the interval connecting $x$ and $y$ is contained in $I_\tau$, i.e. $I_\tau$ contains any interval with boundary points in $I_\tau$. Therefore, $I_\tau = (\alpha_\tau, \beta_\tau)$. The number of distinct intervals in the decomposition (4.2) is at most countable as may may choose a rational point in each of the intervals. □

EXAMPLE 4.29. The complement of the closed interval $[a, b] \subset \mathbf{R}$ is the open subset $(-\infty, a) \cup (b, \infty)$, the union of two disjoint open intervals.

COROLLARY 4.30. Every closed subset $F \subset \mathbf{R}$ is obtained by removing a finite or countably infinite collection of pairwise disjoint open intervals.

## 4.8. The Cantor set

The Cantor set $C \subset [0,1]$ is a closed subset with remarkable properties.

We construct the Cantor set inductively as follows. Let

$$F_0 = [0,1]$$

and let $F_1 \subset F_0$ be obtained by removing the open interval $(1/3, 2/3)$, i.e.

$$F_1 = [0,1/3] \cup [2/3,1].$$

On the next step we remove the middle third from each of the obtained intervals, i.e.

$$F_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,9/9].$$

We continue similarly further and remove the middle third of each of the closed intervals obtained on the previous step. Each set $F_k$ is the union of $2^k$ closed intervals $[\alpha_k, \beta_k]$, each of length $3^{-k}$.

Thus, we obtain an infinite nested sequence of closed sets $F_0 \supset F_1 \supset F_2 \supset \dots$. The Cantor set

$$C = \cap_{n\geq 1} F_n$$

is the intersection of all these closed sets.

THEOREM 4.31. *(a) The Cantor set $C$ has cardinality of continuum and (b) The Lebesgue measure of $C$ is* $0$.

PROOF. To prove (a) we shall use the ternary expansions of real numbers

$$x = 0.a_1a_2a_3\dots, \quad \text{where} \quad a_1, a_2, \dots \in \{0,1,2\}.$$

Each such symbol (i.e. an infinite sequence of digits $0,1,2$) represents the real number

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \dots$$

which is finite and lies in the interval $[0,1]$:

$$x \leq \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \dots = \frac{2}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\right) = \frac{2}{3} \cdot \frac{1}{1-\frac{1}{3}} = 1.$$

□

EXAMPLE 4.32. Let's show that $0.02020202\dots = 1/4$. Indeed,

$$0.02020202\dots = \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \dots = \frac{2}{9} \cdot \left[1 + \frac{1}{9} + \frac{1}{9^2} + \dots\right] = \frac{2}{9} \cdot \frac{1}{1-\frac{1}{9}} = \frac{1}{4}.$$

Some real numbers admit two different ternary representations, for example the expansion

$$x = 0.01222222\dots$$

represents the number

$$0.02 = \frac{2}{9}.$$

Indeed,

$$x = 1/9 + 2/27 \cdot [1 + (1/3) + (1/3)^2 + \dots] = 2/9.$$

**Exercise:** Show that only rational numbers of the form $x = \frac{a}{b}$ with $a$ and $b$ integers and $b$ a power of 3 have two different ternary expansions. All other numbers have a unique ternary expansion.

LEMMA 4.33. *A point $x \in [0,1]$ belongs to the Cantor set $C \subset [0,1]$ if and only if it admits a ternary expansion without 1, i.e. with the symbols $0, 2$ only.*

PROOF. This follows from the way the Cantor set $C$ is constructed: on the step $k$ we delete the numbers having the digit 1 on the position $k$. □

Thus, we obtain a map from the set $S$ of all sequences of symbols $\{0,2\}$ onto the Cantor set $C$. This map becomes bijective once we remove from $S$ a countable subset consisting of sequences having infinite tails of the digit 2. This proves that $C$ has cardinality of continuum, i.e. the statement (a) Theorem 4.31.

To prove the statement (b) of Theorem 4.31 we count the total measure of the intervals removed from $[0,1]$. On the first step we removed an interval of length $1/3$, on the second step we removed 2 intervals of length $1/3^2$, and in general on step $n$ we remove $2^{n-1}$ intervals of length $3^{-n}$. Summing up

$$\begin{aligned} & 1/3 + 2/9 + \frac{4}{27} + \dots + \frac{2^{n-1}}{3^n} + \dots \\ = \; & \frac{1}{2} \cdot [\frac{2}{3} + \frac{4}{9} + \dots + \frac{2^n}{3^n} + \dots] \\ = \; & \frac{1}{2} \cdot \frac{2/3}{1 - 2/3} = 1, \end{aligned}$$

we find that the total measure removed from the interval $[0,1]$ equals 1.