MTH6127 **Metric Spaces and Topology**

Course work 3 **9 February 2024**

1. Let (X, d) be a metric space, and let $\sigma : X \times X \to \mathbb{R}$ be the metric on X defined by $\sigma(x, y) = \min\{d(x, y), 1\}$. Compare the open balls $B^d(c; r)$ and $B^\sigma(c; r)$ with respect to the metrics d and σ.

2. Show that a subset $U \subset X$ is open with respect to the metric d if and only if it is open with respect to σ (defined above).

3. Let (X, d) be a metric space such that for all $x, y \in X$ with $x \neq y$ one has $d(x, y) \geq 1$. Describe in this metric the open balls and open and closed sets.

4. Show that a subset $U \subset \mathbb{R}^m$ is open (closed) with respect to the metric d_p, where $p \in [1, \infty]$, if and only if it is open (closed) with respect to the metric d_∞.

5. For any $p > 0$ we may define a function $||\cdot||_p : \mathbb{R}^m \to \mathbb{R}$ by the usual formula

$$||v||_p = \left[\sum_{i=1}^{m} |x_i|^p\right]^{1/p}, \quad v = (x_1, x_2, \dots, x_m)$$

   Show that this function does not satisfy the triangle inequality for $p \in (0, 1)$ if $m > 1$.

6. Let $d(x, y) = |x - y|$ be the standard metric on the real line $\mathbb{R}$ and let $\sigma(x, y) = \min\{d(x, y), 1\}$ as above. Consider the set $\mathbb{R}^\omega$ of all infinite sequences $\mathbf{x} = (x_n)$ of real numbers $x_n \in \mathbb{R}$, where $n = 1, 2, \dots$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\omega$, where $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, define

$$D(\mathbf{x}, \mathbf{y}) = \sup_{n \geq 1} \left\{ \frac{\sigma(x_n, y_n)}{n} \right\}.$$

   Show that D is well-defined and is a metric on $\mathbb{R}^\omega$.