CHAPTER 4

Topology of metric spaces

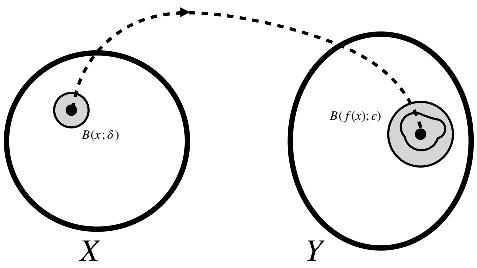
4.1. Continuous maps between metric spaces

Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f : X \to Y$ be a map. The following definition mimics the standard $\epsilon - \delta$ definition of continuity of functions of real variable.

Definition 4.1. We shall say that the map $f : X \to Y$ is continuous at a point $x_0 \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in X$ satisfying $d_X(x, x_0) \leq \delta$ one has $d_Y(f(x), f(x_0)) < \epsilon$.

We can rephrase this definition using the concept of a ball:

Definition 4.2. A map $f : X \to Y$ is continuous at a point $x_0 \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that $f(B(x_0; \delta)) \subset B(f(x_0); \epsilon)$.



Recall that $B(x_0; \delta)$ stands for an open ball with centre x_0 and radius δ.
Instead of $f(B(x_0; \delta)) \subset B(f(x_0); \epsilon)$ one may equivalently write $B(x_0; \delta) \subset f^{-1}(B(f(x_0); \epsilon))$.

Definition 4.3. We shall say that a map $f : X \to Y$ between metric spaces is continuous if it is continuous at every point $x_0 \in X$.

Definition 4.4. A map $f : X \to Y$ between metric spaces is a homeomorphism if it is continuous, bijective and its inverse $f^{-1} : Y \to X$ is also continuous.

Example 4.5. Any interval $[a, b]$ is homeomorphic to $[0, 1]$. Indeed, $f : [0, 1] \to [a, b]$ given by $f(x) = a + (b - a) \cdot x$ is a continuous bijective map and its inverse $f^{-1}(y) = (b - a)^{-1}(y - a)$, $f^{-1} : [a, b] \to [0, 1]$ is also continuous.
Similarly, any half open interval $(a, b]$ is homeomorphic to $(0, 1]$ and to $[0, 1)$.
Any open interval (a, b) is homeomorphic to $(0, 1)$.

Example 4.6. The open interval $(-1, 1)$ is homeomorphic to the real line $\mathbf{R}$. Indeed, we may define $f : (-1, 1) \to \mathbf{R}$ by $f(x) = \tan(\frac{\pi x}{2})$. This function $f : (-1, 1) \to \mathbf{R}$ is bijective and its inverse function $g : \mathbf{R} \to (-1, 1)$ is given by $g(y) = \frac{2}{\pi} \cdot \tan^{-1}(y)$; clearly f and g are continuous.

EXAMPLE 4.7. An open interval $(0, 1)$ is not homeomorphic to the closed interval $[0, 1]$. Indeed, from previous courses we know that any continuous function $f : [0, 1] \to (0, 1)$ attains its maximum and minimum; hence there exist no continuous functions $f : [0, 1] \to (0, 1)$ which are surjective.

Similarly, $(0, 1)$ is not homeomorphic to $(0, 1]$ (exercise).

Homeomorphism is an equivalence relation between metric spaces: it is symmetric, reflexsive and transitive.

If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms then their composition $g \circ f : X \to Z$ is also a homeomorphism.

4.2. Open and closed subsets of metric spaces

Let (X, d) be a metric space.

DEFINITION 4.8. A subset $U \subset X$ is open if for any $x \in U$ there exists $\epsilon > 0$ such that $B(x; \epsilon) \subset U$.

EXAMPLE 4.9. An open interval $(a, b) \subset \mathbf{R}$ is open. Indeed, if $x \in (a, b)$, we can take $\epsilon = \frac{1}{2}\min\{|x - a|, |x - b|\}$. Then $B(x; \epsilon) = (x - \epsilon, x + \epsilon) \subset (a, b)$.

The half open interval $(a, b]$ is not open since there exists no $\epsilon > 0$ such that $B(b; \epsilon) \subset (a, b]$.

LEMMA 4.10. *In any metric space (X, d), an open ball $B(c; r) \subset X$ is open.*

PROOF. Let $x \in B(c; r)$, i.e. $d(x, c) < r$. Take $\epsilon = r - d(x, c) > 0$ and consider the ball $B(x, \epsilon)$. If $y \in B(x, \epsilon)$ then $d(y, c) \leq d(y, x) + d(x, c) = d(y, x) + r - \epsilon < r$. Hence we see that the open ball $B(x, \epsilon)$ is contained in $B(c; r)$. □

The main properties of the family of open subsets of a metric space are summarised in the Lemma below:

LEMMA 4.11. *In a metric space (X, d),*

(O1) *Any union of open subsets of X is open;*
(O2) *Any finite intersection of open subsets of X is open;*
(O3) *The sets $U = X$ and $U = \emptyset$ are open.*

PROOF. (O1) Let $U_\alpha \subset X$ be a family of open subsets where $\alpha \in A$. Consider their union $U = \cup_{\alpha \in A} U_\alpha$. If $x \in U$ then $x \in U_\alpha$ for some $\alpha \in A$. Since U_α is open, there is $\epsilon > 0$ with the property $B(x; \epsilon) \subset U_\alpha$. Then clearly $B(x; \epsilon) \subset U$, i.e. U is open.

(O2) Consider now finitely many open subsets $U_1, U_2, \ldots, U_k \subset X$ and let $U = \cap_{i=1}^{k}$ be their intersection. If $x \in U$ then $x \in U_i$ for every $i = 1, 2, \ldots, k$. Since U_i is open we may find $\epsilon_i > 0$ such that $B(x; \epsilon_i) \subset U_i$. Then, taking $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$, we shall have $B(x; \epsilon) \subset U$, i.e. U is open.

(O3) is obvious. □

THEOREM 4.12. *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is continuous if and only if for any open subset $U \subset Y$ the preimage $f^{-1}(U) \subset X$ is open in X.*

PROOF. Assume that $f : X \to Y$ is continuous according with the Definition 4.3. Let $U \subset Y$ be open. To show that $f^{-1}(U) \subset X$ is open, assume that $x \in f^{-1}(U)$, i.e. $f(x) \in U$. Since U is open, there exists $\epsilon > 0$ such that $B(f(x); \epsilon) \subset U$. Then, according to Definition 4.3, we may find $\delta > 0$ with the property $f(B(x; \delta)) \subset B(f(x); \epsilon)$, i.e. $B(x; \delta) \subset f^{-1}(B(f(x); \epsilon)) \subset f^{-1}(U)$. Hence, $f^{-1}(U)$ is open.

Conversely, suppose that $f : X \to Y$ is such that for any open set $U \subset Y$ the preimage $f^{-1}(U) \subset X$ is open. We want to show that f is continuous as defined in Definition 4.3. For any $x \in X$ and for any $\epsilon > 0$ the ball $B(f(x); \epsilon) \subset Y$ is open. Therefore, its preimage $f^{-1}(B(f(x); \epsilon)) \subset X$ is open. Since $x \in f^{-1}(B(f(x); \epsilon))$, there exists $\delta > 0$ with $B(x; \delta) \subset f^{-1}(B(f(x); \epsilon))$, or equivalently, $f(B(x; \delta)) \subset (B(f(x); \epsilon))$. Therefore, f is continuous. ☐

Next we define closed subsets of a metric space.

DEFINITION 4.13. Let (X, d) be a metric space. A subset $F \subset X$ is said to be closed if its complement $F^c = X - F$ is open. Equivalently, $F \subset X$ is closed if for any $x \in X - F$ there exists $\epsilon > 0$ such that $B(x; \epsilon) \cap F = \emptyset$.

LEMMA 4.14. *In a metric space* (X, d)*,*

(F1) *Any intersection of closed subsets of* X *is closed;*
(F2) *Any finite union of closed subsets of* X *is closed;*
(F3) *The sets* $F = X$ *and* $F = \emptyset$ *are closed.*

PROOF. (F1) Let $F_\alpha \subset X$, where $\alpha \in A$, be a family of closed subsets. Consider their intersection $F = \cap_{\alpha \in A} F_\alpha$. If $x \notin F$ then $x \notin F_\alpha$ for some $\alpha \in A$. Since F_α is closed, there exists $\epsilon > 0$ with $B(x; \epsilon) \cap F_\alpha = \emptyset$. Then $B(x; \epsilon) \cap F = \emptyset$.

(F2) Let $F_1, F_2, \dots, F_k \subset X$ be closed subsets. Consider their union $F = \cup_{i=1}^k F_i$. If $x \notin F$ then $x \notin F_i$ for all $i = 1, 2, \dots, k$. Hence, for any $i = 1, \dots, k$ there is $\epsilon_i > 0$ such that $B(x; \epsilon_i) \cap F_i = \emptyset$. Taking $\epsilon = \min\{\epsilon_1, \dots \epsilon_k\}$ we shall have $B(x; \epsilon) \cap F_i = \emptyset$ for $i = 1, 2, \dots, k$ and hence $B(x; \epsilon) \cap F = \emptyset$.

The property (F3) is obvious. ☐

THEOREM 4.15. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A map* $f : X \to Y$ *is continuous if and only if for any closed subset* $F \subset Y$ *the preimage* $f^{-1}(F) \subset X$ *is closed in* X*.*

PROOF. This is an obvious corollary of Theorem 4.12. Indeed, since

$$f^{-1}(Y - F) = X - f^{-1}(F),$$

we see that the preimage of a closed subset is closed if and only if the preimage of its complement is open. ☐

4.3. The closure of a set

Let (X, d) be a metric space.

For a subset $M \subset X$, define $\overline{M} = \cap F$ where $F \subset X$ runs over all closed subsets containing M. The set $\overline{M}$ is called the closure of M. It is the smallest closed subset of X containing M.

LEMMA 4.16. *Let* (X, d) *be a metric space and let* $M \subset X$ *be a subset.*

(C1) *A point* $x \in X$ *belongs to the closure* $\overline{M}$ *if and only if* $B(x; \epsilon) \cap M \neq \emptyset$ *for any* $\epsilon > 0$*;*
(C2) $M \subset \overline{M}$ *and* $M = \overline{M}$ *if and only if* M *is closed;*
(C3) $\overline{\overline{M}} = \overline{M}$.
(C4) *If* $M_1 \subset M_2$ *then* $\overline{M}_1 \subset \overline{M}_2$.
(C5) $\overline{M_1 \cup M_2} = \overline{M}_1 \cup \overline{M}_2$.

PROOF. (C5) We have $M_1 \subset \overline{M}_1$ and $M_2 \subset \overline{M}_2$ implying $M_1 \cup M_2 \subset \overline{M}_1 \cup \overline{M}_2$. Since the set $\overline{M}_1 \cup \overline{M}_2$ is closed we get $\overline{M_1 \cup M_2} \subset \overline{M}_1 \cup \overline{M}_2$.

On the other hand, we have $M_1 \subset M_1 \cup M_2$ and hence $\overline{M}_1 \subset \overline{M_1 \cup M_2}$ and similarly, $\overline{M}_2 \subset \overline{M_1 \cup M_2}$. This gives $\overline{M}_1 \cup \overline{M}_2 \subset \overline{M_1 \cup M_2}$. ☐

4.4. Convergent sequences in metric spaces

Let $x_1, x_2, \dots$ be a sequence of points $x_n \in X$ of a metric space (X, d).

Definition 4.17. We say that the sequence $\{x_n\}$ converges to a point $x_0 \in X$ if for any $\epsilon > 0$ there exists N such that for all $n > N$ one has $d(x_n, x_0) < \epsilon$. We write $x_0 = \lim x_n$. Equivalently, the sequence $\{x_n\}$ converges to $x_0 \in X$ if for any $\epsilon > 0$ there exists N such that for all $n > N$ one has $x_n \in B(x_0; \epsilon)$.

Example 4.18. Let $X = \mathbf{R}$ with the usual metric. The sequence $x_n = \frac{1}{n}$ converges to $0 \in \mathbf{R}$. The sequences $y_n = (-1)^n$ and $z_n = n$ have no limit.

Lemma 4.19. *If a sequence $x_n \in X$ converges then its limit is unique.*

Proof. Suppose the contrary, i.e. x_n converges to $x_0 \in X$ and to $x_0' \in X$ where $x_0 \neq x_0'$. Take $\epsilon > 0$ so small that $2\epsilon < d(x_0, x_0')$. Then $B(x_0; \epsilon) \cap B(x_0'; \epsilon) = \emptyset$ as follows from the triangle inequality, and for large n the point x_n must lie in both balls $B(x_0; \epsilon)$ and $B(x_0; \epsilon)$ -contradiction. ☐

Lemma 4.20. *A subset $F \subset X$ is closed if and only if for every convergent sequence of points $x_n \in F$ the limit point x_0 also belongs to F.*

Proof. Suppose that $F \subset X$ is closed and $x_n \in F$ converges to a point $x_0 \in X - F$. Then for some $\epsilon > 0$ the ball $B(x_0; \epsilon)$ is disjoint from F. However, if $x_n \to x_0$, we must have $x_n \in B(x_0; \epsilon)$ for all sufficiently large n. This contradicts the assumption that $x_n \in F$.

Suppose that F is not closed, i.e. $F \neq \overline{F}$ and $\overline{F} - F \neq \emptyset$. Let $x_0 \in \overline{F} - F$. Then x_0 belongs to any closed subset containing F. For $\epsilon > 0$, the intersection $B(x_0; \epsilon) \cap F$ is nonempty since otherwise we would have $F \subset X - B(x_0; \epsilon)$ - a closed set not containing x_0. Thus we see that $B(x_0; \frac{1}{n}) \cap F \neq \emptyset$. We may choose $x_n \in B(x_0; \frac{1}{n}) \cap F$ and thus $x_n \to x_0$. Hence the sequence of points $x_n \in F$ has $x_0 \notin F$ as its limit. ☐

4.5. Limit Points

Definition 4.21. Let (X, d) be a metric space. A point $x_0 \in X$ is a limit point of a subset $F \subset X$ if every open ball $B(x_0; \epsilon)$ contains a point of F distinct from x_0.

Example 4.22. If $X = \mathbf{R}$ and $F = (0, 1]$ then every point of the closed interval $[0, 1]$ is a limit point of F. The set $\{n^{-1}; n = 1, 2, \dots\}$ has a unique limit point $0 \in \mathbf{R}$. The set $\mathbf{Z} \subset \mathbf{R}$ has no limit points.

Theorem 4.23. *For a subset F of a metric space X denote by $F' \subset X$ the set of all limit points of F. Then,*

$$\overline{F} = F \cup F'. \tag{4.1}$$

Proof. Suppose that $x_0 \in F'$ and $x_0 \notin \overline{F}$. Since $\overline{F}$ is closed, for some $\epsilon > 0$ one has $B(x_0; \epsilon) \cap \overline{F} = \emptyset$ which contradicts the assumption that x_0 is a limit point of F. We see that $F' \subset \overline{F}$ and hence $F \cup F' \subset \overline{F}$. The argument of the proof of Lemma 4.20 shows that every point $x \in \overline{F} - F$ is a limit point of F, i.e. $x \in F'$. Thus, $\overline{F} \subset F \cup F'$ and (4.1) follows. ☐

Corollary 4.24. A subset of a metric space is closed if it contains all its limit points.

Proof. A subset $F \subset X$ is closed iff $\overline{F} = F$ and the latter holds if and only if $F' \subset F$, as follows from Theorem 4.23. ☐