CHAPTER 2

Balls in metric spaces

2.1. Open and closed balls

Let (X, d) be a metric space. Given a point $c \in X$ and a positive real number $r > 0$.

DEFINITION 2.1. A closed ball with centre c and radius r is defined as the set of all points $x \in X$ such that $d(x, c) \leq r$:

$$B[c; r] = \{x \in X; d(x, c) \leq r\}.$$

An open ball $B(c; r)$ with centre c and radius r is defined similarly:

$$B(c; r) = \{x \in X; d(x, c) < r\}.$$

Clearly $B(c; r) \subset B[c; r]$. The closed ball $B[c; r]$ contains also the points $x \in X$ satisfying $d(x, r) = r$; the latter set can be viewed as *the sphere with centre c and radius r*.

EXAMPLE 2.2. Consider the case $X = \mathbf{R}$ with the usual metric $d(x, y) = |x - y|$. One has

$$B[c; r] = [c - r, c + r],$$

$$B(c; r) = (c - r, c + r).$$

EXAMPLE 2.3. In the case of the plane $X = \mathbf{R}^2$ with the Euclidean metric

$$d((x_1, y_1), (x_2, y_2)) = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{1/2}$$

the ball $B[c; r]$ is

$$\{(x, y) \in \mathbf{R}^2; (x - c_1)^2 + (y - c_2)^2 \leq r^2\},$$

the usual Euclidean ball.

EXAMPLE 2.4. Consider the Hamming metric of §1.2. In the case of the Hamming metric the ball $B[c; r]$ is the set of all words $w \in \Sigma^n$ of length n in the alphabet Σ which are obtained from a fixed word $w_0 \in \Sigma^n$ by making at most r alterations.

LEMMA 2.5. *In a metric space (X, d) the intersection of two closed balls $B[c; r] \cap B[c'; r'] = \emptyset$ is empty if $d(c, c') > r + r'$. Similarly, $B(c; r) \cap B(c'; r') = \emptyset$ if $d(c, c') \geq r + r'$.*

PROOF. If the intersection $B[c; r] \cap B[c'; r']$ is not empty then for $x \in B[c; r] \cap B[c'; r']$ one would have $d(c, x) \leq r$ and $d(x, c') \leq r'$. Using the triangle inequality, we obtain

$$d(c, c') \leq d(c, x) + d(x, c') \leq r + r'.$$

Hence the intersection is empty if $d(c, c') > r + r'$.

The second statement follows similarly. □

We may immediately apply this result to the Hamming metric (Σ^n, d). Let $C \subset \Sigma^n$ be the set of all "code words". As in §1.2 define $\delta = \delta(C)$ (the code distance) to be the minimum of the following set

$$\{d(w_1, w_2) \in \mathbf{R}; w_1, w_2 \in C, w_1 \neq w_2\}.$$

By Lemma 1.2 for a word $w \in \Sigma^n$ there exists at most one code word $v \in C$ which is obtained from w by less than $\delta/2$ alterations. This means that we may successfully correct all errors if our communication channel produces less that $\delta/2$ errors.

2.2. The ball in d_1-metric

As am example we consider in this section the plane $\mathbf{R}^2$ equipped with the d_1-metric. Our goal is to understand the shape of the ball $B[c; r]$.

Assume that the centre c has coordinates $c = (c_1, c_2)$. A point $(x, y) \in \mathbf{R}^2$ lies in the ball $B[c; r]$ iff

$$|x_1 - c_1| + |x_2 - c_2| \leq r. \tag{2.1}$$

Using the definition of the absolute value we see that set of solutions of the inequality (2.1) can be represented as the union of 4 systems of inequalities:

$$x_1 \geq c_1, \quad x_2 \geq c_2, \quad x_1 + x_2 \leq r + c_1 + c_2, \tag{2.2}$$

$$x_1 \geq c_1, \quad x_2 \leq c_2, \quad x_1 - x_2 \leq r + c_1 - c_2, \tag{2.3}$$

$$x_1 \leq c_1, \quad x_2 \geq c_2, \quad -x_1 + x_2 \leq r - c_1 - c_2, \tag{2.4}$$

$$x_1 \leq c_1, \quad x_2 \leq c_2, \quad -x_1 - x_2 \leq r - c_1 - c_2, \tag{2.5}$$

Each of these systems defines a triangle and the union of these 4 triangles has diamond shape as shown on Figure 1. For comparison we consider also the drawings of the balls in d_2 and d_∞ metrics:

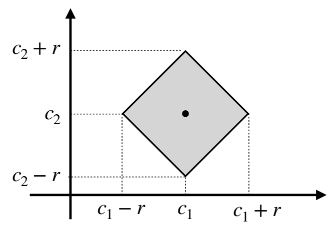


FIGURE 1. Ball $B[c; r]$ in the d_1-metric

We observe that for $p = 1$ and $p = \infty$ the balls for d_p have sharp corners.

On the contrary, one may show that for $p \in (1, \infty)$ the the balls for d_p-metric has smooth boundary.

Placing all three balls together (see Figure 4) we see that d_1-ball sits inside of the d_2-ball and the d_2-ball is inside of d_∞-ball.

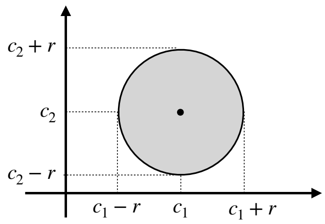


FIGURE 2. Ball $B[c;r]$ in the d_2-metric

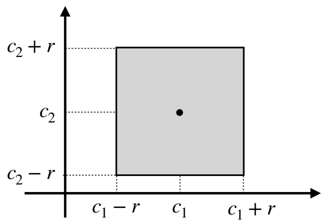


FIGURE 3. Ball $B[c;r]$ in the d_∞-metric

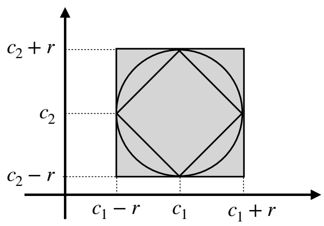


FIGURE 4. Balls in d_1, d_2 and d_∞ metrics

2.3. d_p and d_∞-balls in $\mathbf{R}^m$

We can compare the d_p and d_∞-metrics on $\mathbf{R}^m$ for any $m \geq 1$ and $p \in [1, \infty)$ as follows. Recall that for $v = (x_1, \ldots, x_m)$ and $w = (y_1, \ldots, y_m)$ we have

$$d_\infty(v, w) = \sup_i |x_i - y_i| \quad \text{and} \quad d_p(v, w) = \left[\sum_{i=1}^m |x_i - y_i|^p\right]^{1/p}.$$

Therefore we have the inequalities

$$d_\infty(v, w) \leq d_p(v, w) \leq m^{1/p} \cdot d_\infty(v, w). \tag{2.6}$$

COROLLARY 2.6. One has the following inclusions

$$B^{d_p}[c; r] \subset B^{d_\infty}[c; r] \subset B^{d_p}[c;\, m^{1/p} \cdot r].$$

Recall (see Exercise 1.10) that for $p \in (0,1)$ the norm $||\cdot||_p$ does not satisfy the Minkowski inequality. In other words, for $p \in (0,1)$ and $m > 1$ the ball $B[c,r] = \{v \in \mathbf{R}^m; ||v||_p \leq r\}$ is not convex. Figure 5 below depicts the ball $B[c;r]$ for $m = 2$ and $p = 1/2$.

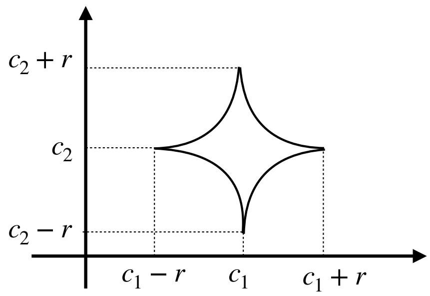


FIGURE 5. "The ball" $|x_1 - c_1|^{1/2} + |x_2 - c_2|^{1/2} \leq 1$.

EXERCISE 2.7. Prove that if (X,d) is a metric space then so is (X,σ) where for $x, y \in X$ one has $\sigma(x,y) = \min\{d(x,y), 1\}$.

EXERCISE 2.8. Compare the open balls $B^d(c;r) \subset X$ and $B^\sigma(c;r) \subset X$ with respect to the metrics d and σ.

CHAPTER 3

Normed and Inner Product Spaces

Here we shall briefly discuss the special classes of normed and inner product spaces, their metric structures and some examples.

3.1. Normed Spaces

Let V be a vector space over $\mathbf{R}$.

DEFINITION 3.1. A norm on V is a function $||\cdot||: V \to \mathbf{R}$ (called "a norm") satisfying the following conditions:

(N1) $||v|| \geq 0$ for any $v \in V$ and $||v|| = 0$ if and only if $v = 0$ (positivity);
(N2) $||\lambda v|| = |\lambda| \cdot ||v||$ for any $\lambda \in \mathbf{R}$ and $v \in V$ (positive homogeneity);
(N3) $||v + w|| \leq ||v|| + ||w||$ (triangle inequality).

DEFINITION 3.2. A normed space is a pair $(V, ||\cdot||)$ where V is a vector space over $\mathbf{R}$ and $||\cdot||$ is a norm on V.

EXAMPLE 3.3. The function $||\cdot||_p : \mathbf{R}^m \to \mathbf{R}$ (which was defined in (1.17)) is a norm for any $p \in [1, \infty]$. The properties (N1) and (N2) are obvious and (N3) for $||\cdot||_p$ is the Minkowski inequality, cf. Theorem 1.8.

EXAMPLE 3.4. Take $V = C[a, b]$ (the space of all continuous functions on the interval $[a, b]$) and let $||\cdot||_{L^1} : V \to \mathbf{R}$ be given by $||f||_{L^1} = \int_a^b |f(x)|dx$. We claim that $||\cdot||_{L^1}$ is indeed a norm.

PROOF. We need to check the properties (N1), (N2), (N3). Clearly, $||f||_{L^1} \geq 0$ and $||f||_{L^1} = 0$ for $f = 0$. We claim that $||f||_{L^1} > 0$ for $f \neq 0$. Indeed, if $f \neq 0$ there is a point $c \in [a, b]$ such that $f(c) \neq 0$. By continuity, there is $\delta > 0$ such that $|f(x)| > |f(c)|/2$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. Thus we see that

$$||f||_{L^1} = \int_a^b |f(x)|dx \geq \delta \cdot |f(c)|/2 \ > \ 0.$$

This concludes the proof of (N1).

The properties (N2) and (N3) follow directly from the corresponding properties of the absolute value. □

EXAMPLE 3.5. Here we briefly mention some other examples of norms on $V = C[a, b]$. Fix a real number $p \in [1, \infty)$ and define $||\cdot||_{L^p} : V \to \mathbf{R}$ by the formula

$$||f||_{L^p} = \left[\int_a^b |f(x)|^p dx\right]^{1/p}, \quad f \in C[a, b].$$

This is similar to the metric d_p of §1.4. We may also define $||\cdot||_{L^\infty} : V \to \mathbf{R}$ by

$$||f||_{L^\infty} = \sup\{f(x); x \in [a, b]\}.$$

We leave the proof of (N1) - (N3) for these norm as an exercise.

EXERCISE 3.6. Show that any ball in a normed space is convex.

3.2. The metric induced by a norm

Let $(V, ||\cdot||)$ be a normed space. Define

$$d : V \times V \to \mathbf{R}$$

by $d(x, y) = ||x - y||$ where $x, y \in V$.

LEMMA 3.7. *The pair (V, d) (as defined above) is a metric space.*

PROOF. We need to check that conditions (M1), (M2), (M3) of Definition 1.1 are satisfied. The positivity $d(x, y) = ||x - y|| \geq 0$ and vanishing only for $x = y$ follow from (N1). Symmetry $d(x, y) = d(y, x)$ follows from (N2) since $d(y, x) = ||y - x|| = ||x - y|| = d(x, y)$. Finally, given three points $x, y, z \in V$ we have using (N3):

$$d(x, z) = ||x - z|| = ||(x - y) + (y - z)|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z).$$

□

EXERCISE 3.8. Let $(V, ||\cdot||)$ be a normed space. For $x, y \in V$, $x \neq y$ define

$$d(x, y) = ||x|| + ||y||$$

and set $d(x, y) = 0$ for $x = y$. Show that $d(x, y)$ is a metric on V. Describe the open balls $B(c, r) \subset V$ in this metric.

3.3. Inner Product Spaces

DEFINITION 3.9. An inner product space is a pair $(V, \langle\, , \rangle)$ where V is a real vector space and $\langle\, , \rangle : V \times V \to \mathbf{R}$ is a function (called an inner product or a scalar product) with the following properties:

(S1) $\langle x, x\rangle \geq 0$ and the equality holds if and only if $x = 0$ (positivity);
(S2) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$ (symmetry);
(S3) $\langle \lambda x + y, z\rangle = \lambda\langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in V$ and $\lambda \in \mathbf{R}$ (linearity).

Note that (S3) only mentions linearity with respect to the first variable only, but symmetry (S2) implies linearity with respect to the second variable as well.

EXAMPLE 3.10. The Euclidean scalar product $\langle\, , \rangle : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ is given by

$$\langle v, w\rangle = \sum_{i=1}^{m} x_i y_i$$

where $v = (x_1, \ldots, x_m)$ and $w = (y_1, \ldots, y_m)$. Properties (S1)-(S3) can be checked directly.

EXAMPLE 3.11. For $V = C[a, b]$ let $\langle\, , \rangle : V \times V \to \mathbf{R}$ be given by

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx. \tag{3.1}$$

It is an inner product on V.

LEMMA 3.12. *In an inner product space $(V, \langle\, , \rangle)$ the following Cauchy inequality is satisfied:*

$$\langle x, y\rangle^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle \quad \textit{for any} \quad x, y \in V. \tag{3.2}$$

PROOF. Using (S1)-(S3) we have

$$0 \leq \langle \lambda x + y, \lambda x + y \rangle = \lambda^2 \cdot \langle x, x \rangle + 2\lambda \cdot \langle x, y \rangle + \langle y, y \rangle.$$

Thus, we see that the quadratic function $\lambda \mapsto \lambda^2 \cdot \langle x, x \rangle + 2\lambda \cdot \langle x, y \rangle + \langle y, y \rangle$ takes only non-negative values. This implies that the discriminant of this quadratic function is non-positive. Hence, we obtain

$$\langle x, y \rangle^2 - \langle x, x \rangle \cdot \langle y, y \rangle \ \leq \ 0$$

which is equivalent to (3.2). □

Any inner product space $(V, \langle \, , \, \rangle)$ defines a normed space $(V, || \cdot ||)$ where the norm $\langle \, , \, \rangle$ is given by

$$||x|| = \sqrt{\langle x, x \rangle}.$$

Properties (N1) and (N2) obviously follow from (S1) and (S2). To check (N3) we shall use the Cauchy inequality (3.2) which can be written in the form

$$\langle x, y \rangle \leq ||x|| \cdot ||y||. \tag{3.3}$$

We have

$$\begin{aligned} ||x + y||^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &\leq ||x||^2 + 2||x|| \cdot ||y|| + ||y||^2 \\ &= (||x|| + ||y||)^2, \end{aligned}$$

i.e. $||x + y|| \leq ||x|| + ||y||$ which is (N3).

3.4. Isometries

Let (X, d_X) and (Y, d_Y) be two metric spaces.

DEFINITION 3.13. The metric spaces (X, d_X) and (Y, d_Y) are said to be *isometric* if there exists a bijection $f : X \to Y$ such that

$$d_Y(f(x), f(y)) = d_X(x, y) \tag{3.4}$$

for all pairs $x, y \in X$.

The bijection satisfying (3.4) is an isometry.

EXAMPLE 3.14. Fix a vector $a \in \mathbf{R}^m$ and consider the map

$$T : \mathbf{R}^m \to \mathbf{R}^m, \quad \text{where} \quad T(v) = v + a, \quad \text{for all} \quad v \in \mathbf{R}^m.$$

This map is known as the parallel translation. Let us show that T is an isometry of $\mathbf{R}^m$ onto itself when viewed with the Euclidean metric

$$d(v, w) = \langle v - w, v - w \rangle^{1/2}.$$

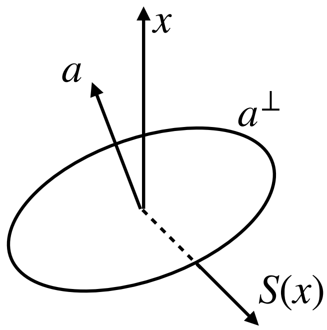
Indeed,

$$d(T(v), T(w)) = \langle v - w, v - w \rangle^{1/2} = d(T(v), T(w)).$$

As another important example of an isometry consider *a reflection* $S : \mathbf{R}^m \to \mathbf{R}^m$ which depends on a choice of a unit vector $a \in \mathbf{R}^m$, i.e. $||a|| = 1$. One defines S by the formula

$$S(v) = v - 2 \cdot \langle v, a \rangle \cdot a.$$

If $v = \lambda a$ then $S(v) = -v$. However, if v is perpendicular to a, i.e. $\langle v, a \rangle = 0$, then $S(v) = v$. Hence S acts as a reflection in the hyperplane perpendicular to the vector a.



Clearly, $S \circ S(v) = v$ for any $v \in \mathbf{R}^m$, i.e. S is an involution. Note also that S is linear and preserves the scalar product:

$$\begin{aligned} \langle S(v), S(w) \rangle &= \langle v - 2\langle v, a \rangle a, w - 2\langle w, a \rangle a \rangle \\ &= \langle v, w \rangle - 4\langle v, a \rangle \cdot \langle w, a \rangle + 4\langle v, a \rangle \cdot \langle w, a \rangle \\ &= \langle v, w \rangle. \end{aligned}$$

We have used above that $\langle a, a \rangle = 1$. Thus we have

$$\begin{aligned} d(S(v), S(w))^2 &= \langle S(v) - S(w), S(v) - S(w) \rangle \\ &= \langle S(v - w), S(v - w) \rangle \\ &= \langle v - w, v - w \rangle \\ &= d(v, w)^2. \end{aligned}$$

This shows that a reflection is an isometry.

THEOREM 3.15 (Cartan - Dieudonne). *Any Euclidean isometry $f : \mathbf{R}^m \to \mathbf{R}^m$ can be represented as a composition*

$$f = T \circ S_k \circ S_{k-1} \circ \cdots \circ S_1, \tag{3.5}$$

where T is a parallel translation and $S_1, S_2, \ldots, S_k$ are reflections. Moreover, such representations exist with $k \leq m$.

Note that representation (3.5) is not unique.