Mathematical Tools for Asset Management MTH6113

Introduction

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Administrative Issues

- Lecturer
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Administrative Issues

- Textbooks
  - Danthine JP, and Donaldson JB, Intermediate Financial Theory, 3rd edition , Academic Press, 2014
  - Hillier D., Grinblatt M., and Titman S, *Financial markets and corporate strategy*, McGraw-Hill Higher Education, 2012
  - Baxter M. and A. Rennie, *Financial Calculus: An Introduction to Derivative Pricing*, 1996
- Lecture Slides posted on QMPlus
- Weekly Problem Sets

Administrative Issues

- Two assessments
  - Assessment 1, Week 6*:* 15% of the final mark
  - Assessment 2, Week 10*:* 15% of the final mark
- Exam: 70% of the final mark

What is this course about?

- **Mathematical Tools for Asset Management**
  - **Financial Economics:** Modelling of financial markets based on economic theory
    - **Choice theory**
  - Subsection:Asset Management
    - **Portfolio Theory and Asset Pricing Models**
    - **Portfolio**: collection of investments (assets) held by an individual or organisation
- What will we study?
  - Economic (and econometric) models which aim to predict asset prices and returns
  - Reference to past data, using time series analysis
    - Stochastic models of security prices

Topics

- Review of Statistics
- The Basis of Economic Theory: Utility Theory
  - Concepts and Optimisation
    - **Rationality**
- Stochastic Dominance
- Portfolio Theory
  - Understanding Risk and Measures of Risks
    - Measures of investment risk
    - Assessment of risk
  - Mean-Variance Analysis

Topics

- Assets Pricing
  - Capital Asset Pricing Model (CAPM)
  - Factor Models and Arbitrage Pricing Theory (APT).
- Efficient Market Hypothesis
  - Comparison and consequences of each form of the hypothesis
  - Testing EMH : Random Walk Model of Asset Pricing
- Introduction to Stochastic Models of Asset Pricing
  - Moving from discrete Random Walks to continuous: Brownian Motions
  - Continuous-time log-normal model
  - Alternative models
- **Bounded Rationality** and Behavioral Finance

Financial Mathematics is applied Mathematics:

- Mathematics is the TOOL to solve real-world problems

This has consequences:

- We will introduce and different financial concepts.
- We need to understand the reality of financial markets and apply economic reasoning!

- If we derive for instance an equation, it is not about the equation itself. The actual meaning lies within its financial interpretation!

Topics

The Market: Large amount of people trading (abstract and concrete) goods on an organised market (exchange), e.g. corn, currencies, stocks, options.

- Not all market participants are human. Most are algorithms!
- Each participant may have different ideas of the value.
  - Yet the market price is where supply and demand meet:
- Prices are determined based on supply and demand.

Portfolio: *collection of investments*

Let's assume stock j part of my portfolio.

Portfolio weight for stock j is the fraction of a portfolio's wealth held in stock j

$$w_j = \frac{\text{Money held in stock } j}{\text{Monetary value of the portfolio}}$$

The sum of all portfolio weights is 1.

Example: A portfolio consists of £1 million in Vodafone stock and £3 million in British Airways stock. What are the portfolio weights of the two stocks?

Answer: The portfolio has a total value of £4 million, hence the fractions of portfolio's wealth held in each stock are:

$$w_{Vodafone} = \frac{1}{4} = 25\%$$
$$w_{BA} = \frac{3}{4} = 75\%$$

In the example: $w_{Vodafone} > 0, w_{BA} > 0$ means that I own positive amounts of both stocks

- **I took a long position** in the two stocks

To **sell short** an asset, the investor sells a security that she doesn't own (borrows it from someone who owns it)

- **take a short position** in a security

To **close out the short position** the investor buys the investment back and returns it to the original owner

- short selling is equivalent with placing a negative weight on a particular stock

A very simple example with short selling

A position with £500,000 in a stock and £100,000 borrowed from a bank has a total investment of £400,000

Stock's weight in the portfolio is $\frac{5}{4} = 1.25$ while the bank investment has a weight of $-\frac{1}{4} = -0.25$

Another example with short selling

An investor takes a long position in British Airways valued at £200,000 and a short position in Vodafone valued at £50,000

Total value of portfolio is £150,000 and hence

$$w_{BA} = \frac{4}{3} = 1.33 > 1$$

$$w_{Vodafone} = -\frac{1}{3} = -0.333 < 0$$

Return on a stock

- For **one share only** the return on that stock is:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

- Note that $P_t = (1 + R_t)\, P_{t-1}$ and hence

$$\ln(1 + R_t) = \ln(P_t) - \ln(P_{t-1})$$

if R_t is small:

$$\ln(1 + R_t) \simeq R_t = \ln(P_t) - \ln(P_{t-1})$$

- Why? Use Taylor series approximation!

One-dimensional Taylor series is an expansion of a real function $f(x)$ about a point $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + ... + \frac{f'^{(n)}(a)}{n!}(x-a)^n$$

Let

$$f(x) = \ln(x+1)$$

We want to find the Taylor series about $x = a = 0$.

Hence:

$$f\left(0\right) = \ln\left(0+1\right) = \ln\left(1\right) = 0$$

$$f'\left(x\right) = \frac{1}{x+1}$$

hence

$$f'\left(0\right) = \frac{1}{0+1} = 1$$

$$f''\left(x\right) = -\frac{1}{\left(x+1\right)^2}$$

hence

$$f''\left(0\right) = -\frac{1}{\left(0+1\right)^2} = -1$$

And so on...

Putting together the series and ignoring the higher order terms we get:

$$\ln(1+x) \cong \ln(1) + \frac{1}{1}(x-0)$$
$$\ln(1+x) \cong x$$

Computing the portfolio return for a two-stock portfolio

Return on an investment: profit divided by the amount invested.

- Example: A portfolio consists of £1 million in Vodafone stock and £3 million in British Airways stock. If Vodafone stock has an annual return of 10% and BA an annual return of 5% what is the annual return of the portfolio?

Answer

Method 1: The end of period value of Vodafone investment: £1.1 million and the end of period value of BA investment £3.15 million

The return of portfolio: $\frac{(1.1+3.15)-4}{4} = 0.0625 = 6.25\%$

Method 2

The return of portfolio

$$w_{Vodafone} \times 0.1 + w_{BA} \times 0.05 = 0.0625 = 6.25\%$$

I could generalize this to N assets

$$R_P = \sum_{i=1}^{N} w_i r_i$$

where R_P is the return of portfolio and r_i return of stock i.

Random variable X- set of possible outcomes x_i from a random experiment

For example in finance the **return on an asset** is in general uncertain and hence a random variable.

- **Discrete random variable**
  - X can take only a countable number of distinct values
  - **the probability distribution** of a discrete random variable X is a list of probabilities associated with each of its possible values:

$$\Pr[X = x_i] = p_i \text{ for any } i$$

where:

$$0 \leq \Pr[X = x_i] \leq 1$$

$$\sum_i p_i = 1$$

- **Continuous random variable**
  - X can take any value and is characterised by a **probability density function (pdf)** $f(x)$, where:

$$\Pr[a \leq X \leq b] = \int_a^b f(x)dx$$

where:

$$f(x) \geq 0 \text{ for all } x$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Cumulative Distribution Function

- For any X **the cumulative distribution function (cdf)** is:

$$F(L) = \sum_{x \leq L} f(x) = \Pr(x \leq L) \text{ discrete case}$$

$$F(L) = \int_{-\infty}^{L} f(x)\,dx = \Pr(x \leq L) \text{ continuous case}$$

where:

$$0 \leq F(L) \leq 1$$

$$\text{if } m > n \text{ then } F(m) > F(n)$$

$$F(+\infty) = 1 \text{ and } F(-\infty) = 0$$

Expectations of a Random Variable

Mean or Expected Value of a Random Variable

$$E(X) \equiv \mu = \sum_i p_i x_i \text{ if } X \text{ is discrete}$$

$$E(X) \equiv \mu = \int_{-\infty}^{\infty} x f(x) dx \text{ if } X \text{ is continuous}$$

Important Results:

$$\begin{aligned} E(aX) &= aE(X) \text{ where } a \text{ is a constant} \\ E(X+Y) &= E(X) + E(Y) \end{aligned}$$

How you relate this to the expected return on a portfolio?

Variance of a Random Variable

$$Var\left(X\right) \equiv \sigma^2 = \sum_i (x_i - \mu)^2 p_i \text{ if } X \text{ is discrete}$$

$$Var\left(X\right) \equiv \sigma^2 = \int_{-\infty}^{\infty} \left(x - \mu\right)^2 f\left(x\right) dx \text{ if } X \text{ is continuous}$$

Standard deviation of a Random Variable

$$Std\left(X\right) \equiv \sigma = \sqrt{Var\left(X\right)}$$

Some Important Results

$$Var\left(X\right) \equiv \sigma^2 = E\left(X^2\right) - \mu^2 \text{ or}$$

$$E\left(X^2\right) = \sigma^2 + \mu^2$$

$$Var\left(a\right) = 0 \text{ where } a \text{ is a constant}$$

$$Var\left(aX\right) = a^2 Var(X)$$

$$Var\left(a + X\right) = Var(X)$$

Joint Distributions

The **joint density function** of two random variables X and Y denoted $f(x, y)$ is:

$$\Pr[a \leq x \leq b, c \leq y \leq d] = \sum_{a}^{b}\sum_{c}^{d} f(x, y) \text{ if } X \text{ and } Y \text{ are discrete}$$

$$\Pr[a \leq x \leq b, c \leq y \leq d] = \int_{a}^{b}\int_{c}^{d} f(x, y)\, dxdy \text{ if } X \text{ and } Y \text{ are continu}$$

The **cumulative distribution function** is

$$F(m, n) = \Pr(X \leq m, Y \leq n)$$

Some important results

$$E[XY] = \sum_{a}^{b} \sum_{c}^{d} xyf(x, y) \text{ if } X \text{ and } Y \text{ are discrete}$$

$$E[XY] = \int_{a}^{b} \int_{c}^{d} xyf(x, y)\, dxdy \text{ if } X \text{ and } Y \text{ are continuous}$$

Independence

If X and Y are full independent we can say that: $F(y|x) = F_Y(y)$

.

Reminder:

$$f(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Hence,

$$f_{X,Y}(x,y) = f_X(x)\, f(y|x)$$

If X and Y independent

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

Mean Independence:$E(X|Y) = E(X)$ and $E(Y|X) = E(Y)$ which implies $E(XY) = E(X)E(Y) \equiv \mu_x \mu_y$

Covariance is defined as:
$Cov(X,Y) \equiv \sigma_{XY} = E\left[\left(X - \mu_x)(Y - \mu_y\right)\right] = E(XY) - \mu_x \mu_y$

If X and Y **independent** then $E(XY) = E(X)E(Y)$ and:

$$Cov(X,Y) \equiv \sigma_{XY} = E(XY) - \mu_x \mu_y = 0$$

The **correlation coefficient** of X and Y is:

$$Corr(X,Y) \equiv \rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Zero Covariance does not imply independence!

Example with two random variables, X and $Y = X^2$
$X \sim U(-1,1)$ with pdf: $f(x) = \frac{1}{1-(-1)} = \frac{1}{2}$.

$$E(X) = \int_{-1}^{1} x f(x)\, dx = \int_{-1}^{1} \frac{x}{2} dx = \left[\frac{x^2}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

$$E\left(X^2\right) = \int_{-1}^{1} x^2 f(x)\, dx = \int_{-1}^{1} \frac{x^2}{2} dx = \left[\frac{x^3}{6}\right]_{-1}^{1} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$\vdots$$

$$E(X^r) = \int_{-1}^{1} x^r f(x)\, dx = \int_{-1}^{1} \frac{x^r}{2} dx = \frac{1}{2}\frac{1}{r+1}\left[x^{r+1}\right]_{-1}^{1}$$

Hence,

$$E\left(X^3\right) = \int_{-1}^{1} x^3 f\left(x\right) dx = \int_{-1}^{1} \frac{x^3}{2} dx = \frac{1}{2}\frac{1}{4}\left[x^4\right]_{-1}^{1} = \frac{1}{8}\left[1^4 - \left(-1\right)^4\right] = 0$$

$$E\left(XY\right) = E\left(X^3\right)$$

$$cov\left(X, Y\right) = E\left(XY\right) - E\left(X\right)E\left(Y\right) = E\left(X^3\right) - E\left(X\right)E\left(X^2\right) = 0 - 0$$

Hence, $cov\left(X, Y\right) = 0$, but X and Y not independent.

Some important results

$$Var\,(aX + bY) = a^2 Var\,(X) + b^2 Var\,(Y) + 2abCov\,(X, Y)$$

If X and Y are **uncorrelated**:

$$\begin{aligned} Var\,(X + Y) &= Var\,(X) + Var\,(Y) \\ Var\,(X - Y) &= Var\,(X) + Var\,(Y) \end{aligned}$$

How you relate this to the variance of returns on a portfolio?

Discrete compounding:

If A is invested for n years at rate R.

The rate is compounded once per year (per annum): at the end of the period: $A\left(1+R\right)^{n}$

The rate is compounded m times per per year (per annum): at the end of the period: $A\left(1+\frac{R}{m}\right)^{mn}$

Continuos compounding: $m \to \infty$

If A is invested for n years at rate R

At the end of the period grows at: $A \exp(Rn)$

Conversion formulae:
R_c : rate of interest with continuos compounding

R_m : the equivalent rate of interest with compounding m times per year

$$A \exp(R_c n) = A \left(1 + \frac{R_m}{m}\right)^{mn}$$

Time Value of Money - Interest Rates

$$\begin{aligned}
\exp(R_c n) &= \left(1+\frac{R_m}{m}\right)^{mn} \\
\ln(\exp(R_c n)) &= \ln\left(1+\frac{R_m}{m}\right)^{mn} \\
R_c n &= mn \ln\left(1+\frac{R_m}{m}\right) \\
R_c &= m \ln\left(1+\frac{R_m}{m}\right)
\end{aligned}$$

Time Value of Money - Interest Rates

$$\exp(\frac{R_c}{m}) = \exp(\ln\left(1+\frac{R_m}{m}\right))$$

$$\exp\left(\frac{R_c}{m}\right) = 1+\frac{R_m}{m}$$

$$R_m = m\left(\exp\left(R_c\,/\,m\right)-1\right)$$