RELATIVITY (MTH 6132)

Module organizers:

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Classes: Monday 11.00–13.00 (Bancroft 1.02.6)
Thursday 10.00–12.00 (Maths MB-204)

The Thursday slot 10.00–11.00 will be a tutorial focused on problem solving and answering questions

Assessment: 80% final exam

20% { 5% week 4 coursework (QM quiz)
10% In-class test on week 8
5% week 11 coursework (QM quiz)

Office hours: Thursday 13.00–14.00 at the learning café, but feel free to get in touch for a 1-2-1 meeting if you prefer

Aims of the course.

Provide a first introduction to two (!) revolutions in our understanding of space and time:

- Special relativity → the concept of spacetime is introduced providing a unique framework for electromagnetism and mechanics
- General relativity: the gravitational force is described by a deformation of spacetime

From the beginning and throughout its history, both phyasists and mathematicians contributed to this story

Physics side	Maths side
• Lorentz	Gauss, Riemann
• Einstein	Poincaré
• Schwarzschild	Grossmann
• Eddington	Hilbert
⋮	⋮

TENTATIVE PLAN OF THE COURSE

week 1 ↔ 3

- Introduction: the Newtonian theory and Galileo's principle of relativity
- Special relativity

week 4 – 10

- Prelude to General Relativity (GR)
- Differential Geometry
- Einstein's equations
- Schwarzschild's exact solution and Black Holes

week 10 – 12

- Gravitational Waves (GWs): introoduction
- Einstein's quadrupole formula
- The binary problem in GR
- Indirect and direct evidence of GWs

Week 1 Monday lecture

PLAN: (1) Kinematics

(2) Newton's mechanics

(3) Newton's law for gravitation

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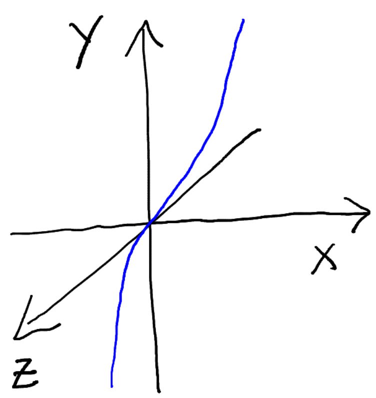
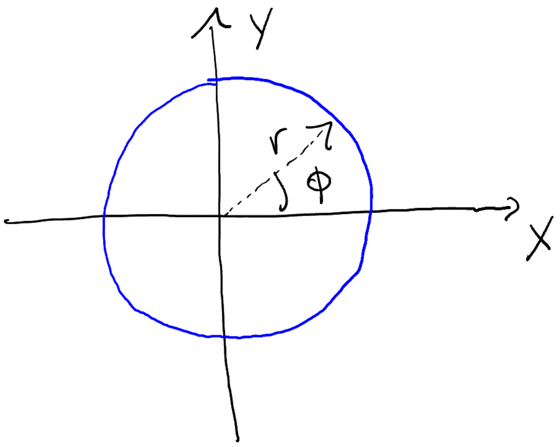
(1) The first thing we need is to be able to describe the motion of objects: this field is called kinematics. In this course we'll deal mainly with the simplest type of objects: point particles. In order to describe them we choose a reference system and then for each instant in time indicate the position of the particles by giving the value of the coordinates at that time. Example 1:



We take a Cartesian system (in black) and for each time give the positions $(x(t), y(t), z(t))$ of the particle. This triplet of functions spans a curve in $\mathbb{R}^3$ (in blue) which is called trajectory.

<u>Example 2:</u> consider a particle moving alon a circle in a plane. We can describe it more easily by using radial coordinate



$$x(t) = r \cos\phi(t)$$
$$y(t) = r \sin\phi(t)$$

The radial system makes it evident that in a circular motion $x^2(t) + y^2(t) = \text{constant in time}$.

<u>Definitions:</u> the time derivative of the position is called the velocity

$$v_x(t) = \frac{dx(t)}{dt}, \quad v_y(t) = \frac{dy(t)}{dt}, \quad v_z(t) = \frac{dz(t)}{dt}$$

Geometrically both the position and the velocity are vectors (in $\mathbb{R}^3$ in the case above) and by construction the velocity is the tangent vector to the trajectory (for each t).

The second time derivative of the position is called the acceleration

$$a_x = \frac{d^2x(t)}{dt^2} = \frac{d v_x(t)}{dt}$$ and similarly for y, z.

(2) The next question is whether we can <u>predict</u> the motion of an object if we know some initial conditions and the forces acting on it.

This area is called <u>dynamics</u>. Let us review the Newtonian theory

- <u>First law</u>: there exist reference systems where an isolated object (i.e. an object on which there is no external influence) keeps moving at constant velocity (which of course can be zero if the object is at rest in the ref. frame).

Such referenc systems are called "Inertial frames"

<u>Example</u>: suppose that a Cartesian system (x, y, z) is an inertial frame and that an isolated

particle is placed at the point $(a, 0, 0)$ with zero velocity. From the first law we can derive the full trajectory at any t

$$x(t) = A, \quad y(t) = z(t) = 0$$

As mentioned we can indicate the position in terms of a vector in $\mathbb{R}^3$ $\underline{x}(t)$

$$\underline{x}(t) = (x(t), y(t), z(t))$$

In the case above $\underline{x}(t) = (A, 0, 0)$.

Second law: Let us work in an inertial frame.
The acceleration of a particle is due to some external force, but what's the precise relation?
According to the second law the force is proportional to the acceleration. In formulae we have $\underline{F} = m\,\underline{a}$ where the constant of proportionality is called "inertial mass" and is indicated by m in the equation above by m.
In Newton's theory m is a constant, so

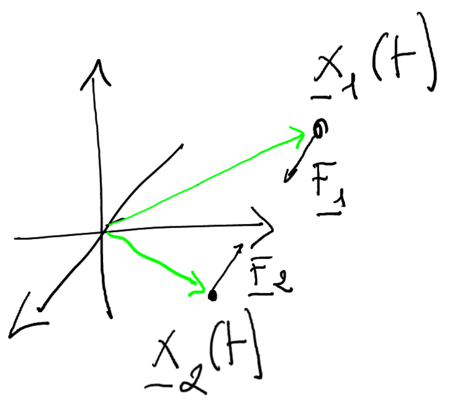
we can rewrite the equation above as

$$\underline{F} = m \frac{d}{dt} \underline{v} = \frac{d}{dt}(m \underline{v})$$

and introduce the concept of (linear) momentum

$$\underline{p} = m \underline{v}$$

Third law: for an isolated inertial system the sum of all forces vanishes. Think for instance at a system of two particles that interact



we have $\underline{F}_1 + \underline{F}_2 = 0$

Theorem: the total momentum of an isolated system in an inertial system is conserved, i.e. it does not change in time. Let me prove for the case in the figure above (the generalisation to an arbitrary number of particles is trivial).

$$\frac{d}{dt}\left(\underline{p}_1 + \underline{p}_2\right) = \frac{d\underline{p}_1}{dt} + \frac{d\underline{p}_2}{dt} \overset{2^{nd}\,law}{=} \underline{F}_1 + \underline{F}_2 \overset{3^{rd}\,law}{=} 0$$

(3) Are there notable examples of forces that we can describe explicitly with an equation.

The main example of interest for this course is Newton's law for the gravitational force

Any object attracts gravitationally all other objects. Let us spell out the formula in the case of point particles. The gravitational force that the first particle exerts on the second is proportional to the product of the two masses and inversely proportional to the square of the separation

$$\underline{F}_2 = \underbrace{G}_{\text{The constant of proportionality is called Newton's constant}} \frac{m_1 m_2}{(\underline{x}_1 - \underline{x}_2)^2} \left(\frac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_1 - \underline{x}_2|}\right)$$

unit vector pointing from the particle 2 to the particle 1

Norm of $\underline{x}_1 - \underline{x}_2$

The force exerted by the particle 2 on the particle 1 is obtained by swapping $1 \leftrightarrow 2$

$$\underline{F}_1 = G \frac{m_1 m_2}{(\underline{x}_2 - \underline{x}_1)^2} \left(\frac{\underline{x}_2 - \underline{x}_1}{|\underline{x}_2 - \underline{x}_1|} \right)$$

and you can easily check that $\underline{F}_1 + \underline{F}_2 = 0$ in agreement with the 3rd law above.

<u>Notable fact 1</u>: conceptually the mass appearing in the formula for the gravitational force could be different from the one appearing in the 2nd law. For this reason sometime you see the concept of gravitational mass being distinguished from that of the inertial mass. The fact that experimentally they are the same was already noticed by Galileo and will play a role in our module

<u>Notable fact 2</u>: it is possible to derive

the gravitational force (a vector) from the gravitational potential U, which is a simpler object since it is a scalar

$$U(\underline{x}_1, \underline{x}_2) = -G\frac{m_1 m_2}{|\underline{x}_1 - \underline{x}_2|} = -G\frac{m_1 m_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$$

and $\underline{F}_1 = -\nabla_{\underline{x}_1} U$ gradient with respect to $\underline{x}_1$

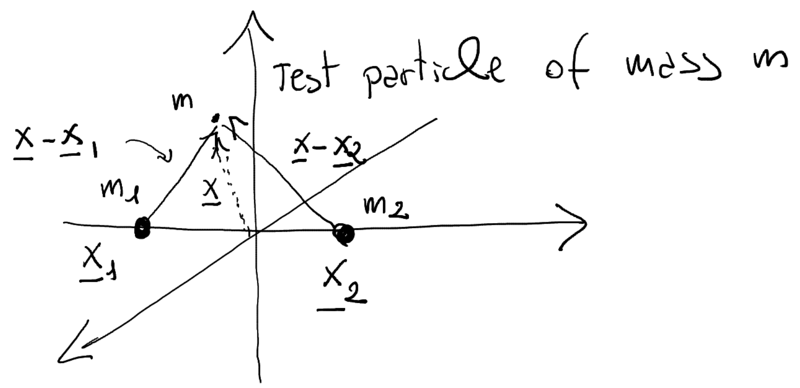
$$(F_1)_x = -\frac{\partial U}{\partial x_1}, \quad (F_1)_y = -\frac{\partial U}{\partial y_1}, \quad (F_1)_z = -\frac{\partial U}{\partial z_1}$$

Check: $$(F_1)_x = \frac{\partial}{\partial x_1}\left(G\frac{m_1 m_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}\right) =$$

$$-G\frac{m_1 m_2 (x_1-x_2)}{\left[(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\right]^{3/2}} = G\frac{m_1 m_2}{|\underline{x}_1-\underline{x}_2|^2}\frac{x_2-x_1}{|\underline{x}_1-\underline{x}_2|}$$

in agreement with the x-component of the formula on the previous page.

<u>Notable fact 3</u>: Newton's theory of gravitation is linear, i.e. the total gravitational force acting on an object is the sum of the gravitational forces exerted by all other objects



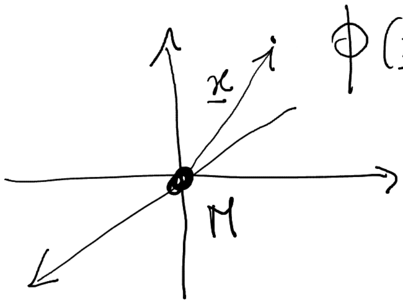
The total gravitational force acting on the test particle is

$$\underline{F}_{TOT} = G \frac{m_1 m}{|\underline{x}_1 - \underline{x}|^2} \frac{\underline{x}_1 - \underline{x}}{|\underline{x}_1 - \underline{x}|} + G \frac{m_2 m}{|\underline{x}_2 - \underline{x}|^2} \frac{\underline{x}_2 - \underline{x}}{|\underline{x}_2 - \underline{x}|}$$

Definition: $\underline{g} = \frac{\underline{F}}{m}$ is the Newtonian gravitational field: it can be defined in each point in space by using a test particle, but notice that m cancels and so $\underline{g}$ depends only on the mass distribution of the system (i.e. the other particles).

From the discussion on the previous page you can see that $\underline{g}$ follows from a scalar function $\phi = \frac{U}{m}$ called gravitational potential

We have $\underset{\sim}{g} = -\nabla_{\underset{\sim}{x}} \phi$ and for a point particle of mass M



$$\phi(\underset{\sim}{x}) = -G\frac{M}{|\underset{\sim}{x}|} = -G\frac{M}{\sqrt{x^2+y^2+z^2}}$$

Notable fact 4: starting from the gravitational potential for a point particle we can check that

$$\nabla^2\phi(x) = \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\phi) = 0 \quad \forall\ \underset{\sim}{x} \neq 0$$

check: $$\nabla^2\phi(x) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}\right]\left(-G\frac{M}{\sqrt{x^2+y^2+z^2}}\right)$$

We have

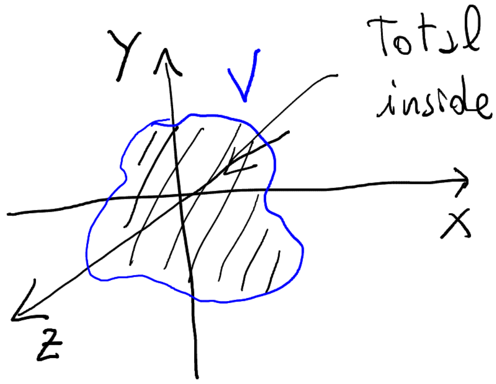
$$\frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{\partial}{\partial x}\frac{2x}{(x^2+y^2+z^2)^{3/2}} =$$

$$-\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[1-\frac{3x^2}{x^2+y^2+z^2}\right] =$$

$$-\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}.$$ By following similar steps you can obtain

$$\nabla^2 \phi(x) = GM \frac{\overset{\text{from } \frac{\partial^2}{\partial x^2}}{(-2x^2+y^2+z^2)} + \overset{\frac{\partial^2}{\partial y^2}}{(x^2-2y^2+z^2)} + \overset{\frac{\partial^2}{\partial z^2}}{(x^2+y^2-2z^2)}}{(x^2+y^2+z^2)^{5/2}} = 0$$

So Newton's gravitational equation in vacuum is simply $\nabla^2\phi = 0$. It is possible to generalise it to the case of a continuous distribution of matter with mass density $\rho(\underline{x})$. This means that $\int_V \rho(\underline{x})\, d^3\underline{x} = \int_V \rho(x,y,z)\, dx\, dy\, dz$ is the total mass inside the region V



Total mass inside $\int_V \rho(\underline{x})\, d^3\underline{x}$

Newton's law

$$\boxed{\nabla^2\phi = 4\pi G \rho}$$

<u>Exercise</u>: Take V to be a spherical ball of radius $|x|$ and the mass density to be concentrated in the origin, i.e. $\rho(\underline{x}) = 0$ if $\underline{x} \neq 0$, but with $\int_V \rho(\underline{x})\, d^3\underline{x} = M$. Start from the equation for $\nabla^2\phi$ and use the divergence theorem to obtain $|\underline{g}| = G\frac{M}{|x|^2}$

Thursday lecture

PLAN for the 1^{st} hour (2^{nd} hour $\to$ TUTORIAL)

(4) Energy conservation

physical units

(5) Galilean relativity

(6) Wave equation (appetizer)

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(4) Let us start from the 2^{nd} law of Newtonian dynamics for a conservative force

$$-\nabla_{\underline{x}} U = \underline{F} = m \frac{d\underline{v}}{dt}$$

and take the scalar product with $\underline{v}$ itself:

$$-\underline{v}\cdot\nabla_{\underline{x}} U = m\,\underline{v}\,\frac{d\underline{v}}{dt} = \frac{1}{2} m^2 \frac{d(\underline{v}\cdot\underline{v})}{dt} = \frac{d}{dt}\left(\frac{1}{2} m \underline{v}^2\right)$$

$$\|$$

$$-\frac{d\underline{x}}{dt}\cdot\nabla_{\underline{x}} U = -\left(\frac{dx}{dt}\frac{\partial U}{\partial x} + \frac{dy}{dt}\frac{\partial U}{\partial y} + \frac{dz}{dt}\frac{\partial U}{\partial z}\right) = -\frac{d}{dt} U(\underline{x})$$

Thus for conservative, time independet forces we have that $\frac{1}{2} m \underline{v}^2 + U$ is conserved

$$\frac{d}{dt}\left[\underbrace{\frac{1}{2} m \underline{v}^2}_{\text{Kinetic energy}} + U(\underline{x})\right] = 0$$

Kinetic energy ↳ Potential energy ⇐ Nomenclature

Important point: You should think about physical quantities (mass, force, energy...) not as pure numbers, as they make sense only if the physical units are specified. We will deal with the following units

QUANTITY	UNIT	
• Length →	metre	m
• time →	second	s
• mass →	Kilogram	Kg

Something I won't discuss: how are these units defined in practice?

Of coure in an equation the l.h.s. and the r.h.s. must have the same physical units. Carrying out this check explicitly is often a very useful consistency check (that goes under the name of "dimensional analysis"].

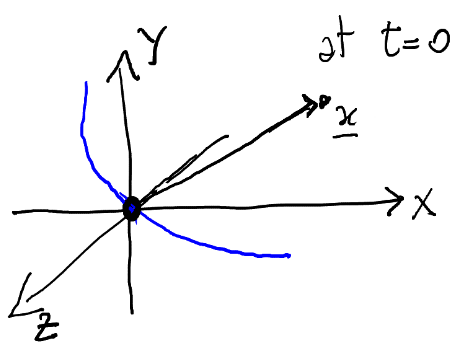
Examples: the unit of the position vector $\underline{x}(t)$ is length ($[\underline{x}(t)] = m$), then $[\underline{v}] = \frac{m}{s}$, $[\underline{a}] = \frac{m}{s^2}$
units

and from $\underline{F} = m\,\underline{a}$, $[\underline{F}] = Kg\,\frac{m}{s^2}$. The gravitational field $[\underline{g}]$ has the same units as the acceleration and for the gravitational potential we have

$$[\underline{g}] = [-\nabla_{\underline{x}}\phi] \Rightarrow \frac{m}{s^2} = \frac{1}{m}[\phi] \Rightarrow [\phi] = \frac{m^2}{s^2}$$

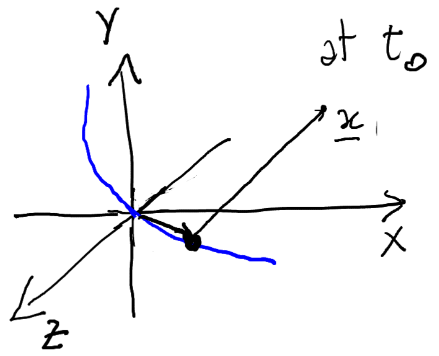
Exercise Show that $[G] = \frac{m^3}{s^2}\,Kg$

(5) A key prediction of Newton's theory is that the gravitational force should be "instantaneous". This means that when a point particle moves the gravitational potential is instantly updated everywhere in space



The particle is at $\underline{x}^{(p)}(t=0) = 0$ and the potential at $\underline{x}$ reads

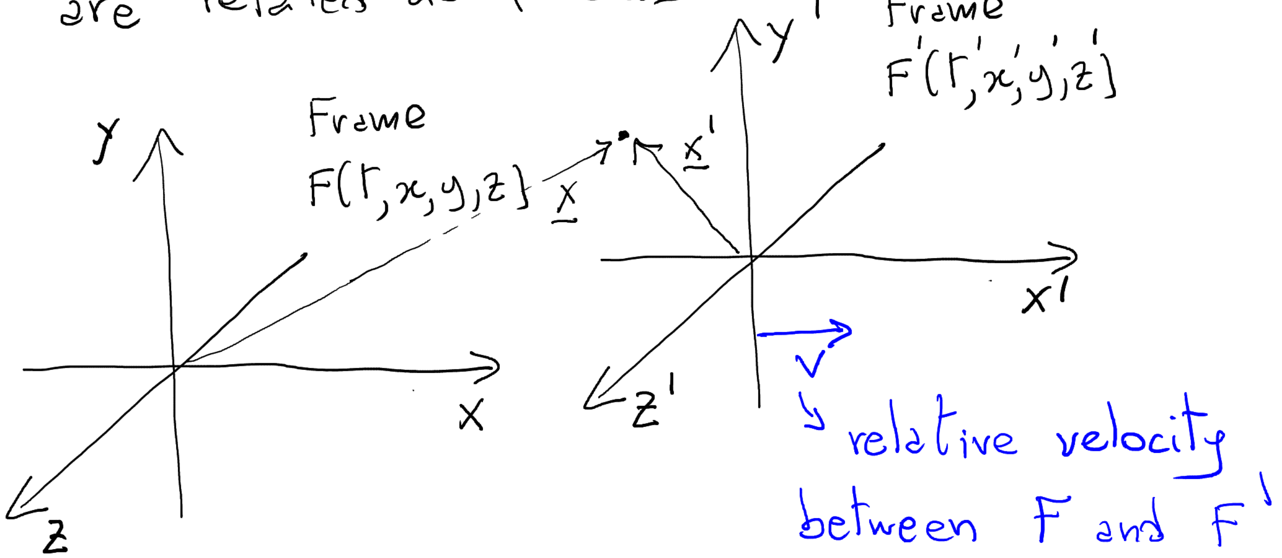
$$\phi(\underline{x}(t=0)) = -G\frac{M}{|\underline{x}|}$$



The particle is at $\underset{\sim}{x}^{(p)}(t_0)$ and the potential at $\underline{x}$ reads

$$\phi(\underline{x}(t=0)) = -G\frac{M}{|\underline{x}-\underline{x}(t_0)|}$$

and this is true for any $\underline{x}$. Then Newtonian physics looks the same in two inertial frames that are related as follows



- $t' = t + a_t$, $\underline{x}' = \underline{x}$ (a_1 is a constant, $\underline{v}=0$)

  time translation (1 parameter)

- $t' = t$, $\underline{x}' = \underline{x} - \underline{v}t$ ($\underset{\sim}{v} = (v_x, v_y, v_z)$ is constant)

  boosts (3 parameters)

- $t' = t$ $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where R is an

  orthogonal 3x3 matrix (3 parameters)

- $t' = t$, $\underline{x}' = \underline{x} + \underline{a}_x$ where $\underline{a}_x$ is a constant vector in $\mathbb{R}^3$ (3 parameters)

<u>Exercise</u>: Write down the equations of motion for a pair of point-like particles interacting via their mutual gravitational force and check that they are invariant under the transformations above.

<u>Nomenclature</u>: These transformations are called "Galilean transformations" and they form a group (in the mathematical sense). Since the dynamical equations are the same in all frames related by these transformations, they are called symmetries.

<u>Comment</u>: Notice that two inertial observers related by a boost would measure two different velocities for the same particle but the same acceleration

$$\underline{x}' = \underline{x} - \underline{v}t \Rightarrow \frac{d\underline{x}'}{dt} = \frac{d\underline{x}}{dt} - \underline{v} \quad (\underline{V}' = \underline{V} - \underline{v})$$

velocity of the particle in F'   velocity of the particle in F   relative velocity between F & F'

However $\dfrac{d^2\underline{x}'}{dt^2} = \dfrac{d^2\underline{x}}{dt^2} - \cancel{\dfrac{d\underline{v}}{dt}} \Rightarrow \underline{a}' = \underline{a}$, since $\underline{v}$ is constant

(6) While I will not cover electromagnetism, the following fact is important for the development of

special relativity: the laws of Maxwell describing how the electromagnetic field behaves it is possible to find in vacuum (i.e. in absence of charges/currents) the following equations should hold

$$\nabla^2 \underline{E} - \epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2} = 0, \quad \nabla^2 \underline{B} - \epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2} = 0$$

where $\underline{E}$ ($\underline{B}$) are the electric (magnetic) fields, ϵ_0 (μ_0) are physical constants ($\epsilon_0 \to$ vacuum permittivity, $\mu_0 \to$ vacuum permeability) which can be measured experimentally $\frac{1}{\sqrt{\epsilon_0 \mu_0}} \equiv c \simeq 3 \times 10^8 \frac{m}{s}$.

The equations above are mathematically the same describing the propagation of waves in a liquid, gas etc. In these physical applications the velocity appearing in the equation does not have a fundamental meaning and is just a property of the medium through which the waves propagate. For instance, sound waves propagate much faster through iron than air. But then what is the interpretation of c?