MTH 4104 Example Sheet III

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III-1. Define the two sets as follows: $S = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, \gcd(a,b) = 1, b \text{ is odd} \right\}$ and $T = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, \gcd(a,b) = 1, b \text{ is non-zero and even} \right\}$. Is S a ring? Is S a field? What about T?

III-2. (a) What is the smallest subset of $\mathbb{R}$ that is a field? (b) What is the smallest subset of $\mathbb{R}$ containing $\sqrt{2}$ that is a field? (c) What is the smallest subset of $\mathbb{R}$ containing $\sqrt{2}$ and $\sqrt{3}$ that is a field?

III-3. Prove the axioms (R$\times$+) and (R+$\times$) for $\mathbb{C}$ (thought of as a ring).

III-4. Let S be the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$. Define addition on S by $(f+g)(x) = f(x) + g(x)$ and multiplication by $(f \times g)(x) = f(g(x))$. Prove that S is not a ring, i.e. find a ring axiom that is not satisfied, by giving a counterexample.

III-5. Given an example of a ring R with an element a such that a is non-zero but $a^2 = 0$ in R.

III-6. Let R the subset $\{[2a]_6 \mid a \in \mathbb{Z}\} = \{[0]_6, [2]_6, [4]_6\}$ of $\mathbb{Z}_6$. Endow R with addition and multiplication that make $\mathbb{Z}_6$ a ring. (a) Does R satisfy the identity law for multiplication, i.e. there exists an element e such that $ae = ea = a$ for every element a of R. Justify your answer. (b) Is R a ring? Is R a field?

III-7. Let n be a positive integer. Write a careful proof for the axiom (R+1) for $\mathbb{Z}_n$.

III-8. Let R be a ring. Prove carefully that $(-a)b = -(ab) = a(-b)$ for all elements a, b of R.

III-9. Let $f = [2]_8 X + [3]_8$ and $g = [4]_8 X^2 + [6]_8 X + [3]_8$ be elements of $\mathbb{Z}_8[X]$. Compute $f + g$ and fg.

III-10. Let F be a field and let f, g be non-zero polynomials in $F[X]$. (a) Is $\deg(fg)$ uniquely determined? If so, what is it? If not, what are the possible values it can take? (b) What about $\deg(f+g)$? (c) What if F is merely a ring?

III-11. Prove the axiom (R$\times$+) for $R[X]$, where R is a ring.

III-12. Given an example of a finite ring R and a function $f : R \to R$ that is not a polynomial (function). Justify your answer.