2. Paths, Cycles, Trees

__Definition__ A __walk__ is an alternating sequence of vertices and edges in which each edge is preceded by one of its endpoints and succeeded by the other. The __length__ of a walk is the number of edges in the sequence. A __u-v-walk__ is a walk that starts at u and ends at v. A walk is __closed__ if it starts and ends with the same vertex. A __trail__ is a walk where all edges are distinct. A __path__ is a trail where all vertices are distinct. A __tour__ is a closed trail. A __cycle__ is a tour that contains at least one edge and in which all vertices except the first and last are distinct. A directed walk (trail, path, tour, cycle) in a digraph is a walk (trail, path, tour, cycle) where each arc is preceded by its tail and succeeded by its head.

2.1 Connectivity

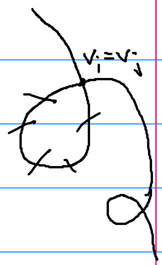
__Definition__ A graph G is __connected__ if for every $u, v \in V(G)$ there is a u-v-walk in G.

__Lemma__ Let G be a graph and $u, v \in V(G)$. Then there exist a u-v-path in G if and only if there exists a u-v-walk in G.

Proof. The direction from left to right is obvious because every path is also a walk.

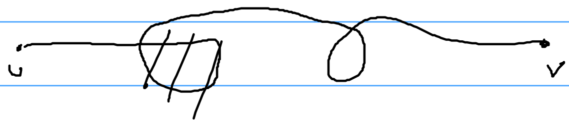
For the direction from right to left, let $w = v_0 e_1 v_1 e_2 v_2 \dots e_n v_m$ be a u-v-walk in G that has minimum length among all u-v-walks in G. Assume for contradiction that w

$$w = v_0 e_1 v_1 e_2 v_2 \dots v_i e_i \dots v_j e_j \dots e_n v_m$$

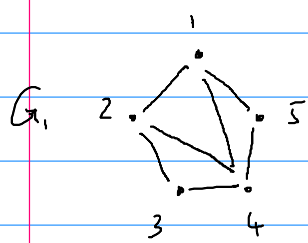


is not a path, i.e., $v_i = v_j$ for $0 \le i < j \le m$. Let $w' = v_0 e_1 v_1 e_2 v_2 \dots v_i e_{j+1} \dots e_m v_m$. Then w' is a u-v-walk that is shorter than w, a contradiction. Therefore w is a u-v-path. $\square$

<u>Corollary</u> A graph G is connected if and only if for every $u, v \in V(G)$ there exists a u-v-path in G.

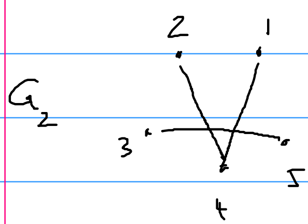


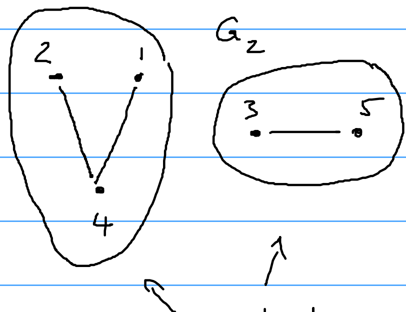
<u>Definition</u> A connected component of a graph G is a maximal connected subgraph of G, i.e., a connected subgraph of G that is not itself a subgraph of a different connected subgraph of G.



G_1 is connected therefore it has a single connected component, G_1 itself.

$1 \text{ — } 2$ is a connected subgraph of G_1; G_1 is a connected subgraph of G_1, and $1 \text{ — } 2$ is a subgraph of that, so $1 \text{ — } 2$ is not a <u>maximal</u> connected subgraph of G_1.



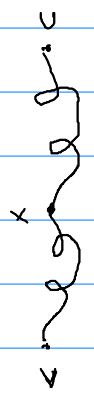


connected components of G_2

Lemma Let G be a graph, G_1 and G_2 two connected subgraphs of G such that $V(G_1) \cap V(G_2) \neq \emptyset$. Then $G_1 \cup G_2$ is connected.

Here, $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

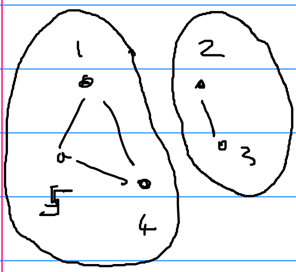
Proof. Let $x \in V(G_1) \cap V(G_2)$, and consider $u, v \in V(G_1 \cup G_2)$. If $u, v \in V(G_1)$, then by connectivity of G_1 there exists a u-v-walk in G_1 and therefore in $G_1 \cup G_2$. If $u, v \in V(G_2)$, then by connectivity of G_2 there exists a u-v-walk in G_2 and therefore in $G_1 \cup G_2$. Finally consider the case where one each of u and v is in G_1 and G_2, and assume without loss of generality that $u \in V(G_1)$ and $v \in V(G_2)$. Since $x \in V(G_1)$, and by connectivity of G_1, there exists a u-x-walk in G_1, and therefore in $G_1 \cup G_2$. Since $x \in V(G_2)$, and by connectivity of G_2, there exists a x-v-walk in G_2, and there $G_1 \cup G_2$. These two walks can be joined to form a u-v-walk in $G_1 \cup G_2$. Therefore $G_1 \cup G_2$ is connected. $\square$



Lemma Consider a graph G with connected components $G_1, G_2, \dots, G_m$. Then $\{V(G_1), V(G_2), \dots, V(G_m)\}$ is a partition of $V(G)$, and $\{E(G_1), E(G_2), \dots, E(G_m)\}$ is a partition of $E(G)$.

Recall: $\{X_1, X_2, \dots, X_m\}$ is a partition of X if $X_i \cap X_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{m} X_i = X$.

Proof. Every vertex of G belongs to some connected component, so $\underline{\bigcup_{i=1}^{m} V(G_i) = V(G)}$.
Now assume for contradiction that $V(G_i) \cap V(G_j) \neq \emptyset$ for $1 \le i < j \le m$. Then by the previous lemma $G_i \cup G_j$ is connected, which contradicts the assumption that G_i and G_j are maximal connected subgraphs of G. Thus $\underline{V(G_i) \cap V(G_j) = \emptyset \text{ for } 1 \le i < j \le m}$, so $\{V(G_1), V(G_2), \ldots, V(G_m)\}$ is a partition of $V(G)$.
Every edge of G belongs to some connected component, so $\underline{\bigcup_{i=1}^{m} E(G_i) = E(G)}$. Now assume for contradiction that $E(G_i) \cap E(G_j) \neq \emptyset$ for some $1 \le i < j \le m$. Let $e \in E(G_i) \cap E(G_j)$, and let v be an endpoint of e. Then $v \in V(G_i) \cap V(G_j)$, which is a contradiction to the fact that $\{V(G_1), V(G_2), \ldots, V(G_m)\}$ is a partition of $V(G)$. Therefore $\underline{E(G_i) \cap E(G_j) = \emptyset \text{ for all } 1 \le i < j \le m}$, so $\{E(G_1), E(G_2), \ldots, E(G_m)\}$ is a partition of $E(G)$. $\square$

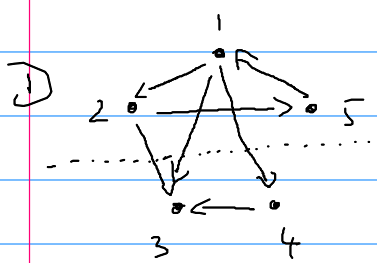


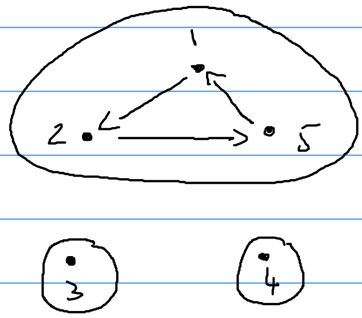
<u>Definition</u> A digraph D is <u>strongly connected</u> if for any $u, v \in V(D)$ there exists a directed u-v-walk in D.

<u>Lemma</u> Let D be a digraph and $u, v \in V(D)$. Then a directed u-v-path in D exists if and only if a directed u-v-walk in D exists.

Corollary A digraph D is strongly connected if and only if for every $u, v \in V(D)$ there exists a u-v-path in D.

Definition A strongly connected component of a digraph D is a maximal strongly connected subgraph of D.



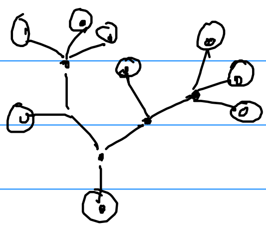


the three strongly connected components of D

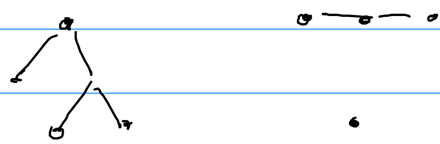
Lemma Consider a digraph D with strongly connected components $D_1, D_2, \dots, D_m$. Then $\{V(D_1), V(D_2), \dots, V(D_m)\}$ is a partition of $V(D)$.

2.2 Trees

Definition A graph G is *acyclic* if it does not contain a cycle, and a *tree* if it is connected an acyclic. A vertex $v \in V(G)$ in a tree G is a leaf if $d_G(v) = 1$.

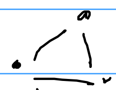


tree with leaves



all trees

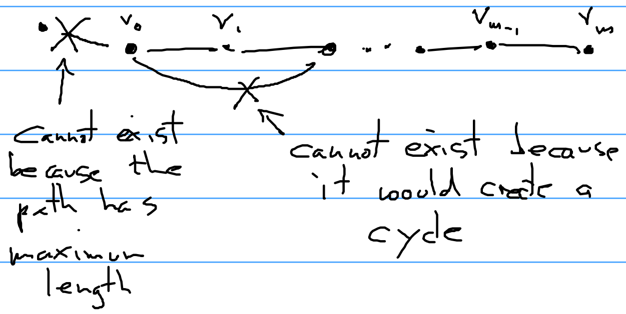
not a tree



not a tree

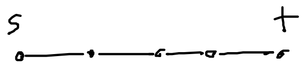
Lemma (tree induction) Every tree with two or more vertices has at least two leaves. Removing any leaf from a tree with n vertices, along with its single incident edge, yields a tree with $n-1$ vertices.

Proof. Consider a tree T with $n \geq 2$ vertices. Consider a path $v_0 v_1 \dots v_{m-1} v_m$ of maximum length among all paths in T. Then $v_0 \neq v_m$ because there are at least two vertices and a path between them. Moreover, since the path has maximum length and T does not contain any cycles, the edge $v_0 v_1$ is the unique edge incident to v_0 in T, and the edge $v_{m-1} v_m$ is the unique edge incident to v_m.



Thus there are at least two leaves in T, v_0 and v_m.

Now consider a leaf v in T, and let u be its unique neighbour. Let T' be the graph obtained by removing v and the edge uv from T. Then T' has $n-1$ vertices. We still need to show that T' is a tree, i.e., connected and acyclic. It is easy to see that T' is acyclic, because it is a subgraph of the acyclic graph T. To see that T' is connected, consider $s, t \in V(T')$. Note that $s, t \in V(T)$, and by connectivity of T, there is an s-t-path in T. Note further that v is not contained in this path: $v \notin V(T')$, so $v \neq s$ and $v \neq t$; any other



vertex in the path has degree at least 2 in T, while v has degree 1. Hence the s-t-path is also contained in T', so T' is connected. So T' is a tree. $\square$

<u>Theorem</u> If T is a tree, then $|E(T)| = |V(T)| - 1$.