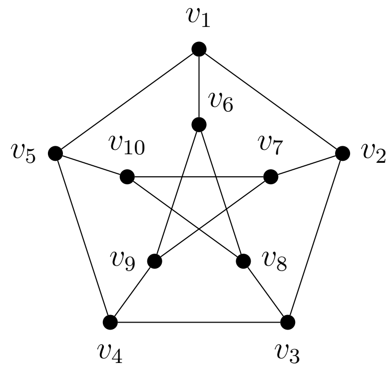
MTH6105 – Algorithmic Graph Theory **Spring 2024**

Problem Sheet 2 **F. Fischer**

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

---

1. In the graph shown below, find (a) a shortest v_1–v_8-path, (b) a longest v_1–v_8-path, (c) a shortest cycle, and (d) a longest cycle. Explain in each case why the path or cycle you have found has minimum or maximum length.



Note: We have not yet discussed any algorithms for finding shortest and longest paths and cycles. You may therefore have to look for them by trial and error, and explain at the end why you are done.

Solution: As the graph is simple, we can write a walk as a sequence of vertices rather than an alternating sequence of vertices and edges. The length of the walk is then equal to the length of the sequence of vertices minus one. Note that the walks described below are not unique, but that their lengths of course are.

(a) A shortest path v_1–v_8-path is $v_1v_6v_8$, which has length 2. There cannot be a shorter path, as v_1 and v_8 are not adjacent.

(b) A longest v_1–v_8-path is $v_1v_6v_9v_7v_2v_3v_4v_5v_{10}v_8$, which has length 9. There cannot be a longer path, as a path may not visit any vertex more than once.

(c) A shortest cycle is $v_1v_2v_3v_4v_5v_1$, which has length 5. We can check that there are no shorter cycles by enumerating all paths of length at most 3 and checking that the first and last vertex on the path are not adjacent.

(d) A longest cycle is $v_1v_2v_3v_4v_5v_{10}v_7v_9v_6v_1$, which has length 9. Assume for contradiction that there was a cycle of length 10, and label the vertices $u_1, u_2, \ldots, u_{10}$ along the cycle. In addition to the 10 edges that form the cycle there must then be 5 more edges between vertices that are not next to each other on the cycle. Each vertex in the graph has degree 3 and must therefore be an endpoint of exactly one of these 5 edges. We already know that there are no cycles of length 4 or less, so none of the 5 edges can be between vertices that have distance less than 4 along the cycle, i.e., between u_i and u_{i+k} for some $k < 4$

and $1 \leq i \leq 10 - k$. It also cannot be that the 5 edges are between all pairs of vertices that have distance 5 along the cycle, i.e., between u_i and u_{i+5} for all $1 \leq i \leq 5$. There thus has to be an edge between two vertices that have distance 4 along the cycle, which without loss of generality we may assume to be the edge u_1u_5. It is now easy to check that the addition of another edge with endpoint u_6 would create a cycle of length at most 4, which we know is a contradiction.

Bonus question: What if instead of a shortest or longest cycle we seek a shortest or longest closed trail?

---

2. (a) Show that any digraph that contains a closed directed walk of length at least one contains a directed cycle.
   (b) What is the analogous statement for graphs? Give a proof or a counterexample for this statement.

Solution:

(a) Let $W = v_0e_1v_1 \ldots e_mv_0$ be a closed directed walk in D that has length at least one and, subject to this condition, is as short as possible. Assume for contradiction that W is not a directed cycle. Then $v_i = v_j$ for some $0 \leq i < j < m$. Let $W' = v_0e_1v_1 \ldots e_iv_ie_{j+1}v_{j+1} \ldots e_mv_0$. Then W' is a closed directed walk in D. It contains the arc e_m and therefore has length at least one, and it is shorter than W. This contradicts the assumption that W has minimum length. Hence W must be a directed cycle.

(b) The analogous statement for graphs, that the existence of a closed walk of length at least one implies the existence of a cycle, is not true. Let G be the simple graph with $V(G) = \{u, v\}$ and $E(G) = \{uv\}$. Then uvu is a closed walk in G, but there is no cycle. Note that the proof from Part a fails for graphs because in graphs there can be a closed walk, like the closed walk uvu above, that repeats an edge but does not repeat a vertex apart from the first one.

---

3. Find all unlabeled trees with six vertices. You may want to start by considering the sequence $d_1, d_2, \ldots, d_6$ of degrees of the vertices in such a tree, and using what you know about this sequence.

Solution: We know that for any tree G, $|E(G)| = |V(G)| - 1$, and for any graph G, $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$. Thus $\sum_{i=1}^{6} d_i = 2(6-1) = 10$. We also know that for all $i \in [6]$, $1 \leq d_i \leq 5$. If we list degrees in non-decreasing order, the following sequences are thus possible:

$$(5,1,1,1,1,1) \quad (4,2,1,1,1,1) \quad (3,3,1,1,1,1) \quad (3,2,2,1,1,1) \quad (2,2,2,2,1,1)$$

We can now see by inspection that there are two distinct unlabeled trees for the fourth sequence, where the vertex with degree 3 respectively has one or two neighbors with degree 1. For all other sequences there is a unique unlabeled tree. We obtain the following six trees: