MTP24, Lecture 1: Basics of Measure Theory

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Introduction

- For ground set Ω how can we *measure* subsets $A \subset \Omega$?

If Ω is finite or countable (e.g. $\{1, \ldots, n\}, \mathbb{N}, \mathbb{Z}$), assign $\mu(\omega) \geq 0$ to each $\omega \in \Omega$, then let

$$\mu(A) := \sum_{\omega \in A} \mu(\omega).$$

for *every* $A \subset \Omega$. This defines on the power set $\mathcal{P}(\Omega)$ a function μ, which is *σ-additive*,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k), \quad \text{for (pairwise) disjoint } A_k$$

nonnegative, $0 \leq \mu(A) \leq \infty$, and satisfies $\mu(\varnothing) = 0$.

- For uncountable spaces $\mathbb{R}, \mathbb{R}^n, \{0,1\}^\infty, \mathbb{R}^\infty, C([0,1])$ we want to consider also more involved measures that may assign positive measure to sets whose individual points receive measure zero.

Fundamental example: the Lebesgue measure on $\mathbb{R}$

$\lambda(I) := b - a$ for interval $I = (a, b], -\infty < a < b < \infty$.
For (pairwise) disjoint intervals $I_1, I_2, \ldots$ let

$$\lambda\left(\bigcup_{k=1}^{\infty} I_k\right) := \sum_{k=1}^{\infty} \lambda(I_k).$$

For single point $\lambda(x) = \lambda(\{x\}) = 0$, thus since $\mathbb{Q}$ is countable also

$$\lambda(\mathbb{Q}) = \sum_{x \in \mathbb{Q}} \lambda(x) = 0,$$

while $\lambda((a, b] \setminus \mathbb{Q}) = b - a$.

However, the Cantor set also has Lebesgue measure 0 but cardinality continuum.

- But there is no good way to define λ for *all* subsets of $\mathbb{R}$.

Example: the 'coin-tossing' space

$\Omega = \{0,1\}^\infty$ models an infinite series of independent Bernoulli trials with probability p for outcome 1 and $1-p$ for outcome 0. Then $\mathbb{P}(A) = (1-p)p$ for $A = \{\omega = (\omega_1, \omega_2, \ldots) : \omega_1 = 0, \omega_0 = 1\}$

To which $A \subset \Omega$ can we assign probability? For instance, what is the probability of

$$A = \{\omega \in \Omega : \lim_{n\to\infty} \frac{\omega_1 + \cdots + \omega_n}{n} = p\}.$$

σ-algebras

Definition A family of sets $\mathcal{F} \subset \mathcal{P}(\Omega)$ is called *σ-algebra* if

(i) $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,

(iii) $A_k \in \mathcal{F} \ \ (k \in \mathbb{N})$ disjoint sets $\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$.

Then $\varnothing \in \mathcal{F}$, and $\mathcal{F}$ is closed under countable intersections.

Definition $(\Omega, \mathcal{F})$ is called *measurable space*, and $A \in \mathcal{F}$ are called ($\mathcal{F}$-)*measurable sets* or *events* in the probability context.

Examples

(i) $\{\varnothing, \Omega\}$ (trivial σ-algebra),

(ii) $\mathcal{P}(\Omega)$ (the power set),

(iii) for partition $\Omega = \cup_{k=1}^{\infty} A_k$ in disjoint nonempty subsets $A_1, A_2, \ldots$, the following system of union-sets is a σ-algebra:

$$\bigcup_{k \in J} A_k, \quad J \subset \mathbb{N}$$

(iv) for any family $\{\mathcal{F}_t,\ t \in T\}$ of σ-algebras (T arbitrary set)

$$\bigcap_{t \in T} \mathcal{F}_t$$

is a σ-algebra,

(v) but the union of σ-algebras $\mathcal{F}_1 \cup \mathcal{F}_2$ typically is not a σ-algebra.

Generators

Definition For $\mathcal{G} \subset \mathcal{P}(\Omega)$, the intersection of all σ-algebras (in Ω) containing $\mathcal{G}$ is called the σ-algebra *generated* by $\mathcal{G}$, denoted $\sigma(\mathcal{G})$. This is the smallest σ-algebra containing $\mathcal{G}$.

If $\mathcal{F} = \sigma(\mathcal{G})$ we call $A \in \mathcal{G}$ *generators* of $\mathcal{F}$. A σ-algebra admitting a countable family of generators is called *separable*.

Example $\Omega = \{0,1\}^\infty$, the 2^k *cylinder* sets

$$A(\epsilon_1, \ldots, \epsilon_n) = \{\omega \in \Omega : \omega_1 = \epsilon_1, \ldots, \omega_n = \epsilon_n\}.$$

generate a finite σ-algebra $\mathcal{F}_n$. We have *filtration* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ The set $A = \{\omega : (\omega_1 + \cdots + \omega_n)/n \text{ converges as } n \to \infty\}$ is not in $\cup_n \mathcal{F}_n$ but $A \in \mathcal{F}$ for $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$.

Borel σ-algebra

The σ-algebra of *Borel* sets in $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$ is generated by any of the families of sets:

(i) open sets,

(ii) closed sets,

(iii) intervals $(a, b]$, with $a, b \in \mathbb{R}$

(iv) intervals $(a, b]$ with $a, b \in \mathbb{Q}$ ($\mathcal{B}(\mathbb{R})$ is separable!),

(v) halflines $(-\infty, b]$, where $b \in \mathbb{R}$ (alternativley, $b \in \mathbb{Q}$)

For topological space $\mathcal{X}$, the Borel σ-algebra $\mathcal{B}(\mathcal{X})$ is the σ-algebra generated by the family of open sets. Examples are $\mathbb{R}^n, \mathbb{R}^\infty, C([a, b])$, etc.

Criteria for σ-algebra: monotone class

- Let $\mathcal{A}$ be an algebra (closed under *finite* unions), and *monotone class*: that is for $A_k \in \mathcal{A}$

$$A_1 \subset A_2 \subset \cdots \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A},$$

$$A_1 \supset A_2 \supset \cdots \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A},$$

then $\mathcal{A}$ is a σ-algebra.

Criteria for σ-algebra: $\pi - \lambda$ system

A family $\mathcal{D}$ of subsets in Ω is called a π-*system*, if closed under finite intersections, that is

$$A_1, A_2 \in \mathcal{D} \Rightarrow A_1 \cap A_2 \in \mathcal{D}.$$

A family $\mathcal{D}$ is called a λ-system if

(i) $\Omega \in \mathcal{D}$,

(ii) $A, B \in \mathcal{D}, A \subset B \Rightarrow B \setminus A \in \mathcal{D}$,

(iii) $A_1 \subset A_2 \subset \cdots, A_n \in \mathcal{D} \Rightarrow \bigcup_{n=1}^{\infty} A_k \in \mathcal{D}$.

[Alternative conditions defining λ-system: (i), (ii') closed under taking complement set and (iii') closed under disjoint countable unions.]

Dynkin's Theorem: a π-λ-system is a σ-algebra.

Definition of a measure

Definition A *measure* on a mesurable space $(\Omega, \mathcal{F})$ is a nonnegative function $\mu : \mathcal{F} \to [0, \infty]$ such that

(i) $\mu(\varnothing) = 0$,

(iii) for disjoint $A_1, A_2, \ldots,$ with $A_n \in \mathcal{F}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

If $\mu(\Omega) < \infty$ the measure is called *finite*. If $\Omega = \cup_{k=1}^{\infty} \Omega_k$ where $\mu(\Omega_k) < \infty$, the measure is σ-*finite*.
If $\mu(\Omega) = 1$ we speak of a *probability measure* and may use notation $\mathbb{P}$.

Criteria for σ-additivity.

(i) Subadditivity: for $A_1, A_2, \ldots,$ with $A_n \in \mathcal{F}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

(iii) Monotonicity (increasing tower): $A_1 \subset A_2 \subset \cdots \Rightarrow$ $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

(iv) Monotonicity (decreasing tower): $A_1 \supset A_2 \supset \cdots \Rightarrow$ $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

(v) Continuity at 'zero': $A_1 \supset A_2 \supset \cdots, \bigcap_{n=1}^{\infty} A_n = \varnothing \Rightarrow \lim_{n\to\infty} \mu(A_n) = 0$.

Uniqueness of measures

Theorem Let $(\Omega, \mathcal{F})$ be a measurable space, where $\mathcal{F} = \sigma(\mathcal{D})$ for a π-system $\mathcal{D}$. Suppose $\Omega = \cup_k \Omega_k$, where $\Omega_k \in \mathcal{D}$. If two measures μ and ν coincide on $\mathcal{D}$ and $\nu(\Omega_k) = \mu(\Omega_k) < \infty$ then $\mu(A) = \nu(A)$ for all $A \in \mathcal{F}$.

Idea of proof: the collection of A's with $\mu(A) = \nu(A)$ is a λ-system, so Dynkin's theorem applies.

Construction by extension

A σ-additive μ_0 on algebra (or other set family) is called *pre-measure*. For instance, the intervals $I = (a, b] \subset \mathbb{R}$ comprise an algebra, and if $I = \cup_{n=1}^{\infty} I_n$ then $\lambda(I) = \sum_{n=1}^{\infty} \lambda(I_n)$.

Caratheodory Theorem: Suppose μ_0 is a pre-measure on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ algebra. Then there exists a measure μ on $(\Omega, \sigma(\mathcal{A})$ such that

$$\mu(A) = \mu_0(A), \quad A \in \mathcal{A}.$$

Such measure μ_0 is unique if for some $\Omega_k \in \mathcal{A}$

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k, \qquad \Omega_1 \subset \Omega_2 \subset \cdots$$

and $\mu_0(\Omega_k) < \infty, k \in \mathbb{N}$.

Examples of extension

- Lebesgue measure on $\mathcal{B}(\mathbb{R})$ is the extension from the algebra of intervals of the function 'interval length'.

- For c.d.f. F on $\mathbb{R}$ there is a unique probability measure with $\mu((-\infty, x]) = F(x), \quad x \in \mathbb{R}$. The measure may have no atoms, but have no density function (example: Cantor ladder).

- Lebesgue measure on $\mathcal{B}(\mathbb{R}^n)$ is the extension from the algebra of n-dimensional intervals $(a_1, b_1] \times \cdots \times (a_n, b_n]$. The intervals $(-\infty, b_1] \times \cdots \times (-\infty, b_n]$ comprise a π-system.

• Bernoulli(p) measure on $\{0,1\}$ is the extension from the algebra 'finite-dimensional' cylinders $A(\epsilon_1,\dots,\epsilon_k)$. The event 'frequency of '1's is p becomes probability 1 (strong Law of Large Numbers).

In the case $p=1/2$ the function

$$\omega \mapsto \sum_{k=1}^{\infty} \frac{\omega_k}{2^k}$$

establishes a measure-theoretic isomorphism between $\{0,1\}^\infty$ with Bernoulli($1/2$) and $[0,1]$ with λ. In the case $p \neq 1/2$ the pushforward of Bernoulli(p) measure is *singular* relative to λ, that is supported by a Borel set of zero Lebesgue measure.

Space $\mathbb{R}^\infty$

The Borel σ-algebra $\mathcal{B}(\mathbb{R}^\infty)$ is generated by finite-dimensional cylinders

$$C(B^n) := \{x \in \mathbb{R}^\infty : (x_1, \ldots, x_n) \in B^n\}, \quad B^n \in \mathcal{B}(\mathbb{R}^n),\ n \in \mathbb{N},$$

where for B^n's we can also take n-dim intervals or smaller family $B^n = (-\infty, b_1] \times \cdots \times (-\infty, b_n]$, $b_i \in \mathbb{R}$ (or $b_i \in \mathbb{Q}$).

Probability measures P_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \in \mathbb{N}$ are called *consistent* if

$$P_{n+1}(B^n \times \mathbb{R}) = P_n(B^n), \quad B^n \in \mathcal{B}(\mathbb{R}^n).$$

Kolmogorov's Extension Theorem: Suppose P_n are consistent probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then there exists a unique probability measure $\mathbb{P}$ on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ such that

$$\mathbb{P}(C(B^n)) = P_n(B^n), \quad B^n \in \mathcal{B}(\mathbb{R}^n), \quad n \in \mathbb{N}.$$

Lebesgue measurable sets

Definition A family of sets $\mathcal{S} \subset \mathcal{P}(\Omega)$ is a *semiring* is $\mathcal{S}$ is closed under finite intersections and

$$A, B \in \mathcal{S} \Rightarrow A \setminus B = C_1 \cup \cdots \cup C_n,$$

for some n and disjoint $C_k \in \mathcal{S}$.
For pre-measure μ on $\mathcal{S}$ define *exterior measure*

$$\mu^*(A) = \inf \sum_{n=1}^{\infty} \mu(A_n)$$

where $A \subset \cup_n A_n$, and $A_n \in \mathcal{S}$ disjoint.
Clearly, $\mu^*(A) = \mu(A)$ for $A \in \mathcal{S}$.

Set $A \subset \Omega$ is called *Lebesgue-measurable* if for every $\epsilon > 0$ there exists $B \in \mathcal{S}$ such that

$$\mu^*(A \Delta B) < \epsilon.$$

Denote this family $L(\mathcal{S}, \mu)$.

Lebesgue's Theorem: The family $L(\mathcal{S}, \mu)$ is a σ-algebra and μ^* is a measure on $L(\mathcal{S}, \mu)$.

Example For $\mathcal{S} = \mathcal{B}(\mathbb{R})$ the σ-algebra of Lebesgue-measurable sets is

$$L(\mathcal{B}(\mathbb{R}), \lambda) = \sigma(\mathcal{B}(\mathbb{R}) \cup \mathcal{N}),$$

where $\mathcal{N}$ is the family of *nullsets* A, such that $A \subset B$ for some B Borel set with $\lambda(B) = 0$.
This sort of measure extension by including all nullsets in the system of generators is called *measure completion*.