MATH 5105 Differential and Integral Analysis : Solution Sheet 10

1. Calculate the following (improper) integrals

   (a) $\int_0^1 \log x dx$,

   Proof. As $\log(x)$ has a singularity at 0, this is an improper integral. Computing

$$\begin{aligned}\int_0^1 \log(x)dx &= \lim_{\varepsilon\to 0}\int_\varepsilon^1 \log x dx \\ &= \lim_{\varepsilon\to 0}\Big[x\log x\Big]_\varepsilon^1 - \int_\varepsilon^1 \frac{x}{x}dx \\ &= 0 - \lim_{\varepsilon\to 0}(\varepsilon\log\varepsilon - [1-\varepsilon]) \\ &= -1\end{aligned}$$

   where we evaluate $\lim_{x\to 0} x\log x = 0$ using L'Hôpital's rule. □

   (b) $\int_2^\infty \frac{\log x}{x} dx$,

   Proof.

$$\begin{aligned}\int_2^\infty \frac{\log x}{x}dx &= \lim_{a\to\infty}\int_2^a \left(\frac{d}{dx}\log x\right)\log x dx \\ &= \lim_{a\to\infty}\Big[(\log x)^2\Big]_2^a - \lim_{a\to\infty}\int_2^a \frac{\log x}{x}dx\end{aligned}$$

   Hence we get that

$$2\int_2^\infty \frac{\log x}{x}dx = \lim_{a\to\infty}\Big[(\log x)^2\Big]_2^a = \infty.$$

   □

   (c) $\int_0^\infty \frac{1}{1+x^2}dx$.

Proof. Let $x = \tan\theta$, $dx = \sec^2\theta d\theta = (1+\tan^2\theta)d\theta = (1+x^2)d\theta$. Furthermore as $x \to \infty$ it follows that $\theta \to \frac{\pi}{2}$ so that

$$\begin{aligned}\lim_{a\to\infty}\int_0^a \frac{1}{1+x^2}dx &= \lim_{\theta_0\to\pi/2}\int_0^{\theta_0}\frac{1+x^2}{1+x^2}d\theta \\ &= \lim_{\theta_0\to\pi/2}\theta_0 = \frac{\pi}{2}.\end{aligned}$$

□

2. Find the radius of convergence and the exact intervals of convergence for the following power series

(a) $\sum n^2 x^n$,

Proof. Let us consider $a_n = n^2$ so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right| = 1 = \beta.$$

Hence $R = \frac{1}{\beta}$ and the radius of convergence ins 1. If $x = \pm 1$ the the series $\sum n^2(-1)^n, \sum n^2(1)^n$ are divergent by the n-th term test. □

(b) $\sum \frac{2^n}{n!}x^n$,

Proof. Let $a_n = \frac{2^n}{n!}$ then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right| \\ &= \lim_{n\to\infty}\left|\frac{2^{n+1}n!}{2^n(n+1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0.\end{aligned}$$

$\therefore \beta = 0 \& R = \frac{1}{\beta} = \infty \implies$ the interval of convergence is $\mathbb{R}$. □

(c) $\sum \frac{3^n}{n4^n}x^n$,

Proof.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)4^{n+1}}\frac{n4^n}{3^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{3}{4}\frac{n}{n+1}\right| \\ = \frac{3}{4} = \beta.\end{aligned}$$

Therefore the radius of convergence is $R = \frac{1}{\beta} = \frac{4}{3}$. If $x = \frac{4}{3}$ then we have

$$\sum \frac{3^n}{n4^n} \frac{4^n}{3^n} = \sum \frac{1}{n}$$

which diverges by the p harmonic test ($p = 1$). If $x = -\frac{4}{3}$ then

$$\sum (-1)^n \frac{3^n}{n4^n} \frac{4^n}{3^n} = \sum \frac{(-1)^n}{n}$$

which converges by the Leibniz test. Therefore the interval of convergence is $\left[-\frac{4}{3}, \frac{4}{3}\right)$. □

(d) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

Proof. Note that we can not use the ratio test as $a_{2n} = 0$ but $a_{2n+1} = \frac{3^n}{\sqrt{n}}$. Instead we treat the sum as an ordinary series and apply the ratio test to consecutive terms

$$\lim_{n\to\infty} \left| \frac{3^{n+1}/\sqrt{n+1} \times x^{2n+3}}{3^n/\sqrt{n} \times x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{3\sqrt{n}}{\sqrt{n+1}} x^2 \right|$$
$$= 3x^2 < 1$$

if $|x| < \frac{1}{\sqrt{3}}$. Testing the end points, as $x = \frac{1}{\sqrt{3}}$ we get

$$\sum \frac{3^n}{\sqrt{n}} 3^{-n-1/2} = \sum \frac{1}{\sqrt{3n}}$$

which diverges by the p harmonic test (with $p = \frac{1}{2}$). As $x = -\frac{1}{\sqrt{3}}$ we get

$$\sum \frac{3^n}{\sqrt{n}} \frac{(-1)^{2n+1}}{\sqrt{3}3^n} = -\sum \frac{1}{\sqrt{3n}}$$

which again diverges by the p-harmonic test. Hence the interval of convergence is

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

□

3. For all $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n} \sin nx$. Each f_n is differentiable. Show that

(a) $\lim_{n\to\infty} f_n(x) = 0$,

Proof. Note that $|\sin nx| \leq 1$ for all n, x hence

$$|f_n(x)| = \left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n}$$

which gives us

$$\lim_{n\to\infty} f_n(x) = 0$$

as $\lim_{n\to\infty} \frac{1}{n} = 0$. □

(b) Show that $\lim_{n\to\infty} f_n'$ may not exist.

Proof. Computing we get

$$f_n'(x) = \frac{n}{n}\cos(nx) = \cos(nx).$$

Therefore if we let $x = \pi$ then

$$f_n'(\pi) = \cos(n\pi) = (-1)^n$$

which is divergent. Hence $\lim_{n\to\infty\to\infty} f_n(x)$ many not exist. □

4. Let $f_n(x) = nx^n, \quad x \in [0,1], n \in \mathbb{N}$

(a) Show that $\lim_{n\to\infty} f_n(x) = 0, \quad x \in [0,1)$,

Proof. If $0 < x < 1$ then we see that $x = 1/r^n$ where $r > 1$

$$f_n(x) = nx^n$$

and we can apply say L'Hôpital's rule to conclude that $\lim_{n\to\infty} f_n(x) = 0$. □

(b) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$.

Proof. Note that the function $f(x)$ is not defined at $x = 1$ but considering it as an improper integral

$$\int_0^1 f(x)dx = \lim_{\varepsilon\to 1}\int_0^{\varepsilon} f(x)dx = 0.$$

On the other hand

$$\begin{aligned}\int_0^1 f_n(x)dx &= \int_0^1 nx^n dx \\ &= \left[\frac{n}{n+1}x^{n+1}\right]_0^1 \\ &= \frac{n}{n+1}\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = 1 \neq \int_0^1 f(x)dx.$$

□

5. By considering $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$, derive the formula

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad |x| < 1$$

Proof. Recall that

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Differentiating with respect to x, we get

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Then multiplying by x we get

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

□

Furthermore

(a) evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$,

Proof. If we set $x = \frac{1}{2}$ (here $|x| < 1$) we get

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2.$$

□

(b) evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}, \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

Proof. If $x = \frac{1}{3}, x = -\frac{1}{3}, (|x| < 1)$ so that we have

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{1/3}{4/9} = \frac{3}{4}$$

and

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{3^n} = \frac{-1/3}{(1+1/3)^2} = -\frac{3}{16}.$$

□

6. (a) Derive an explicit formula for

$$\sum_{n=1}^{\infty} n^2 x^n$$

Proof. Recall that

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Differentiating with respect to x, we get

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}.$$

Then multiplying by x we get

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

Again differentiating with respect to x

$$\begin{aligned}\sum_{n=1}^{\infty} n^2 x^{n-1} &= \frac{1}{(1-x)^2} - \frac{-2x}{(1-x)^3} \\ &= \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}.\end{aligned}$$

Therefore we have

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}.$$

□

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}, \sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

Proof. If $x = \frac{1}{2}$, then

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1/2 \times 3/2}{(1-1/2)^3} = \frac{3 \times 8}{4} = 6.$$

Similarly if $x = \frac{1}{3}$ then

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{1/3 \times 4/3}{(1-1/3)^3} = \frac{4/9}{(2/3)^3} = \frac{3}{2}.$$

□

7. Let $f(x) = |x|, x \in \mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$? Explain your answer.

Proof. No. Because if such a power series existed then f would be smooth but f is not differentiable at $x = 0$.

□

8. Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sum_{k=1}^{\infty} \sin^2\left(\frac{x}{k}\right)$ differentiable?

Proof. If $\sum f_k$ converges (uniformly) and $\sum f_k'$ converges uniformly then $f = \sum f_k$ is differentiable and $f' = \sum f_k'$.
Therefore let $f_k(x) = \sin^2\left(\frac{x}{k}\right)$. Then $f_k'(x) = \frac{2}{k}\sin\left(\frac{x}{k}\right)\cos\left(\frac{x}{k}\right)$. As $|\sin t| \leq |t|$ for all $t \in \mathbb{R}$, we have

$$\sum_{k=1}^{\infty} \left|\sin^2\left(\frac{x}{k}\right)\right| \leq |x|^2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since $\sum \frac{1}{k^2}$ converges, the sum$\sum_{k=1}^{\infty} f_k$ converges for fixed x. This implies that $\sum_{k=1}^{\infty} f_k$ converges pointwise. Also as $|\cos t| \leq 1$ for all $t \in \mathbb{R}$, we have

$$\left|2\sin\left(\frac{x}{k}\right)\cos\left(\frac{x}{k}\right)\right| \leq 2|x|\frac{1}{k^2} \quad \forall k \in \mathbb{N}.$$

If we restrict x to $[-A, A]$ for some $A > 0$ then for all $k \in \mathbb{N}$,

$$\left|2\sin\left(\frac{x}{k}\right)\cos\left(\frac{x}{k}\right)\right| \leq 2|x|\frac{1}{k^2} \leq 2A\frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} \frac{2A}{k^2} = 2A\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the WeierstraßM-test implies that $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly on $[-A, A]$. □

9. Let $\{f_n\}$ be a sequence of integrable functions on $[a, b]$ and suppose that $f_n \to f$ uniformly on $[a, b]$. Prove that f is integrable and that

$$\int_a^b f dx = \lim_{n\to\infty} \int_a^b f_n.$$

10. Let $f_n : [0, \infty) \to \mathbb{R}$ be a sequence of continuous functions that converge uniformly to $f(x) = 0$. Show that if

$$0 \le f_n(x) \le e^{-x}$$

for all $x \ge 0$ and for all $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = 0.$$

Proof. As f_n converges uniformly to 0, we have that

$$\lim_{n\to\infty} \int_0^M f_n(x)dx = 0$$

for any fixed $M > 0$. To deal with the improper integral, we split the integral $\int_0^\infty f_n(x)dx$ into two pieces by writing

$$\int_0^\infty f_n(x)dx = \int_0^M f_n(x)dx + \int_M^\infty f_n(x)dx.$$

As we are dealing with improper integrals, we need to be precise with the limits involved, so we write for $A > M$

$$\int_0^A f_n(x)dx = \int_0^M f_n(x)dx + \int_M^A f_n(x)dx.$$

We have therefore

$$\begin{aligned}\lim_{n\to\infty} \int_0^\infty f_n(x)dx &= \lim_{n\to\infty} \lim_{A\to\infty} \left(\int_0^M f_n(x)dx + \int_M^A f_n(x)dx \right) \\ &= \lim_{n\to\infty} \int_0^M f_n(x)dx + \lim_{n\to\infty} \lim_{A\to\infty} \int_M^A f_n(x)dx \\ &= \lim_{n\to\infty} \lim_{A\to\infty} \int_M^A f_n(x)dx.\end{aligned}$$

Now by assumption $0 \le f_n(x) \le e^{-x}$ and therefore

$$0 \le \int_M^A f_n(x)dx \le \int_M^A e^{-x}dx \le e^{-M}.$$

thus

$$0 \leq \lim_{n\to\infty} \lim_{A\to\infty} \int_M^A f_n(x)dx \leq e^{-M}.$$

This holds for any choice of $M > 0$, so we obtain the upper bound $\inf_{M>0} e^{-M} = 0$. □