MATH 5105 Differential and Integral Analysis
Exercise Sheet 10

Problems

1. Calculate the following integrals

   (a) $\int_0^1 \log x dx$,

   (b) $\int_2^\infty \frac{\log x}{x} dx$,

   (c) $\int_0^\infty \frac{1}{1+x^2} dx$.

2. Find the radius of convergence and the exact intervals of convergence for the following power series

   (a) $\sum n^2 x^n$,

   (b) $\sum \frac{2^n}{n!} x^n$,

   (c) $\sum \frac{3^n}{n4^n} x^n$,

   (d) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

3. For all $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n} \sin nx$. Each f_n is differentiable. Show that

   (a) $\lim_{n\to\infty} f_n(x) = 0$,

   (b) Show that $\lim_{n\to\infty} f_n'$ may not exist.

4. Let $f_n(x) = nx^n, \quad x \in [0,1], n \in \mathbb{N}$

   (a) Show that $\lim_{n\to\infty} f_n(x) = 0, \quad x \in [0,1)$,

   (b) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$.

5. Observe that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad |x| < 1$$

   (a) Evaluate $\sum_{n=1}^\infty \frac{n}{2^n}$,

   (b) Evaluate $\sum_{n=1}^\infty \frac{n}{3^n}, \sum_{n=1}^\infty \frac{(-1)^n n}{3^n}$.

6. (a) Derive an explicit formula for

$$\sum_{n=1}^{\infty} n^2 x^n$$

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}, \sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

7. Let $f(x) = |x|, x \in \mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$? Explain your answer.

9*. Let $\{f_n\}$ be a sequence of integrable functions on $[a, b]$ and suppose that $f_n \to f$ uniformly on $[a, b]$. Prove that f is integrable and that

$$\int_a^b f dx = \lim_{n\to\infty} \int_a^b f_n.$$

10*. Let $f_n : [0, \infty) \to \mathbb{R}$ be a sequence of continuous functions that converge uniformly to $f(x) = 0$. Show that if

$$0 \le f_n(x) \le e^{-x}$$

for all $x \ge 0$ and for all $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x) dx = 0.$$

11*. Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sum_{k=1}^{\infty} \sin^2\left(\frac{x}{k}\right)$ differentiable?