MATH 5105 Differential and Integral Analysis
Assignment 5 Solutions

1. Let $f_n(x) = \frac{1+\cos^{2023}(nx)}{\sqrt{n}}$. Does $\{f_n\}$ converge uniformly to a function f on $\mathbb{R}$?

Proof. Note that for a fixed x we see that $\lim_{n\to\infty} f_n(x) = 0$ pointwise. It remains to prove that the convergence is uniform. Hence consider

$$|f_n(x) - 0| = |f_n(x)| = \left|\frac{1+\cos^{2023}(nx)}{\sqrt{n}}\right| \leq \left|\frac{2}{\sqrt{n}}\right|$$

and $\frac{2}{\sqrt{n}} < \varepsilon$ if $\frac{2}{\varepsilon} < \sqrt{n}$ or equivalently $n > \frac{4}{\varepsilon^2}$. Therefore we choose

$$N(\varepsilon) = \left\lceil \frac{4}{\varepsilon^2} \right\rceil.$$

□

2. (a) Show for all $x \in \mathbb{R}$, the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos\left(\frac{x}{n}\right)$ converges uniformly.

Proof. As $|\cos t| \leq 1$ for all $t \in \mathbb{R}$, we have

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^2}\cos\left(\frac{x}{n}\right)\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^2},$$

As $\sum \frac{1}{n^2}$ converges, the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos\left(\frac{x}{n}\right)$ converges (absolutely) for fixed x as absolute convergence implies convergence. □

(b) Show for all $x \in \mathbb{R}$, the sum $\sum_{n=1}^{\infty} \frac{1}{n^3}\sin\left(\frac{x}{n}\right)$ converges uniformly.

Proof. As $|\sin t| \leq 1$ for all $t \in \mathbb{R}$, we have

$$\left|\frac{1}{n^3}\sin\left(\frac{x}{n}\right)\right| \leq \frac{1}{n^3}$$

for all $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} \frac{1}{n^3}\sin\left(\frac{x}{n}\right)$ converges uniformly by the Weierstraß M-test. □

3. Does the pointwise limit function of the $\sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{x}{n}\right)$ exist as a function $f : \mathbb{R} \to \mathbb{R}$, i.e. for all $x \in \mathbb{R}$? If the function f exists, decide whether or not the convergence is uniform.

   Proof. We note that substituting $x = 0$ in $\sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{x}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent series and therefore the pointwise limit function f is not defined at $x = 0$. Therefore, all other issues of convergence, and uniform convergence cannot be considered. □

4. Find a relationship between $\frac{1}{n^2} \cos\left(\frac{x}{n}\right)$ and $\frac{1}{n^3} \sin\left(\frac{x}{n}\right)$ and use this to show $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right)$$

   is differentiable.

   Proof. Let $f_n = \frac{1}{n^2} \cos\left(\frac{x}{n}\right)$. Then $f_n'(x) = -\frac{1}{n^3} \sin\left(\frac{x}{n}\right)$. As $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise and as $g(x) = \sum_{n=1}^{\infty} f_n'(x)$ converges uniformly we can integrate term by term to get

$$\int_0^x -g(t)dt = \sum_{n=1}^{\infty} \int_0^x -\frac{1}{n^3} \sin\left(\frac{t}{n}\right) dt = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right) = f(x).$$

   As g is continuous, the fundamental theorem of calculus shows is that f is differentiable and that

$$f'(x) = -g(x) = -\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{x}{n}\right)$$

□