MTH5105 Differential and Integral Analysis 2012-2013

Solutions 9

1 Exercises

1) (a) Show that for all $x \in \mathbb{R}$, the sum $\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$ converges.
[You may use that $|\sin(t)| \leq |t|$ for all $t \in \mathbb{R}$.]

(b) Show that the sum $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$ converges uniformly for all $x \in \mathbb{R}$.

(c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$$

is differentiable.

Solution:

(a) As $|\sin t| \leq |t|$ for all $t \in \mathbb{R}$, we have

$$\sum_{k=1}^{\infty} \left| \frac{1}{k} \sin\left(\frac{x}{k}\right) \right| \leq |x| \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$\sum \frac{1}{k^2}$ converges, so that the sum $\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$ converges absolutely (for fixed x). (Recall absolute convergence implies convergence.)

(b) As $|\cos t| \leq 1$ for all $t \in \mathbb{R}$, we have

$$\left| \frac{1}{k^2} \cos\left(\frac{x}{k}\right) \right| \leq \frac{1}{k^2} .$$

for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges the sum $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$ converges uniformly by the Weierstraß M-test.

(c) Let $f_k(x) = \frac{1}{k} \sin\left(\frac{x}{k}\right)$. Then $f_k'(x) = \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$.
As $\sum f_k$ converges pointwise and $\sum f_k'$ converges uniformly, $f = \sum f_k$ is differentiable and $f' = \sum f_k'$. (This is by Theorem 9.8(b) in the notes.)

2) Is the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sum_{k=1}^{\infty} \sin^2(x/k)$$

differentiable?

Solution:

If $\sum f_k$ converges pointwise and $\sum f_k'$ converges uniformly, then $f = \sum f_k$ is differentiable and $f' = \sum f_k'$.

Let $f_k(x) = \sin^2(x/k)$. Then $f_k'(x) = 2\sin(x/k)\cos(x/k)/k$.

As $|\sin t| \le |t|$ for all $t \in \mathbb{R}$, we have

$$\sum_{k=1}^{\infty} |\sin^2(x/k)| \le x^2 \sum_{k=1}^{\infty} \frac{1}{k^2} .$$

$\sum \frac{1}{k^2}$ converges, so that the sum $\sum_{k=1}^{\infty} f_k$ converges absolutely (for fixed x). This implies $\sum_{k=1}^{\infty} f_k$ converges pointwise

[We could have proven uniform convergence on bounded intervals, but we don't need to.]

As also $|\cos t| \le 1$ for all $t \in \mathbb{R}$, we have

$$|2\sin(x/k)\cos(x/k)/k| \le |x|\frac{1}{k^2}$$

for any $k \in \mathbb{N}$.

If we restrict x to $[-A, A]$ for some $A > 0$, then, for all $k \in \mathbb{N}$,

$$|2\sin(x/k)\cos(x/k)/k| \le |x|\frac{1}{k^2} \le A\frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} A\frac{1}{k^2} = A\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the Weierstraß M-test implies that $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly on $[-A, A]$.

Thus we can conclude that $f = \sum f_k$ is differentiable with $f' = \sum f_k'$ on any interval $[-A, A]$, and hence on $\mathbb{R}$.

3) Let $f_n : [0,1] \mapsto \mathbb{R}$ be a sequence of differentiable functions, and let $f : [0,1] \mapsto \mathbb{R}$. Consider the statements

(a) $f_n \to f$ pointwise,

(b) $f_n \to f$ uniformly,

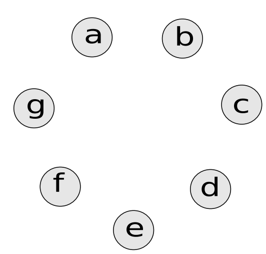
(c) f_n' converges pointwise,

(d) $f_n' \to f'$ pointwise,

(e) f continuous,

(f) f differentiable,

(g) $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$,



and cleary indicate in the enclosed figure all implications by the appropriate arrows ("$\Longrightarrow$").

Solution:

The only valid implications are:

(b) implies (a),(e),(g)

(d) implies (c)

(f) implies (e)

*4) Let $f_n : [0,\infty) \mapsto \mathbb{R}$ be a sequence of continuous functions that converge uniformly to $f(x) = 0$. Show that if

$$0 \leq f_n(x) \leq e^{-x}$$

for all $x \geq 0$ and for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = 0 \ .$$

[Recall from Calculus I the definition of the improper integral $\int_0^\infty f(x)\,dx = \lim_{A\to\infty} \int_0^A f(x)\,dx$.]

Solution:

As f_n converges uniformly to zero, we have that

$$\lim_{n\to\infty} \int_0^M f_n(x)\,dx = 0$$

for any fixed $M > 0$.

To deal with the improper integral, we split the integral $\int_0^\infty f_n(x)\,dx$ into two pieces by writing

$$\int_0^\infty f_n(x)\,dx = \int_0^M f_n(x)\,dx + \int_M^\infty f_n(x)\,dx \ .$$

As we are dealing with improper integrals, we need to be precise with the limits involved, so we write for $A > M$

$$\int_0^A f_n(x)\,dx = \int_0^M f_n(x)\,dx + \int_M^A f_n(x)\,dx \ ,$$

and take the appropriate limit of $A \to \infty$.

We have therefore

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx &= \lim_{n\to\infty} \lim_{A\to\infty} \left(\int_0^M f_n(x)\,dx + \int_M^A f_n(x)\,dx \right) \\
&= \lim_{n\to\infty} \int_0^M f_n(x)\,dx + \lim_{n\to\infty} \lim_{A\to\infty} \int_M^A f_n(x)\,dx \\
&= \lim_{n\to\infty} \lim_{A\to\infty} \int_M^A f_n(x)\,dx \ .
\end{aligned}$$

Now by assumption $0 \leq f_n(x) \leq e^{-x}$, and therefore

$$0 \leq \int_M^A f_n(x)\,dx \leq \int_M^A e^{-x}\,dx < e^{-M} \ .$$

Thus

$$0 \leq \lim_{n\to\infty} \lim_{A\to\infty} \int_M^A f_n(x)\,dx \leq e^{-M} \ .$$

This holds for any chosen $M > 0$, whence the upper bound is $\inf_{M>0}(e^{-M}) = 0$.